# 16.901 作业#4 答案

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## 欧拉方程

**1** 静态压力 $P$ 为

$$P = \rho R T$$

然而，指定能量与 $T$ 的关系或许是：

$$e = c_v T$$

$$\Rightarrow P = \frac{R}{c_v} \rho e$$

回忆有 $R = c_p - c_v$ $\gamma = c_p / c_v$，

$$\Rightarrow \frac{R}{c_v} = \frac{c_p}{c_v} - 1 = \gamma - 1$$

$$\Rightarrow P = \frac{R}{c_v} \rho e = (\gamma - 1) \rho e$$

最后，因为 $E = e + \frac{1}{2} u^2 \Rightarrow P = (\gamma - 1) \rho \left( E - \frac{1}{2} u^2 \right)$ 或者

$$\boxed{P = (\gamma - 1) \left( \rho E - \frac{1}{2} \rho u^2 \right)}$$

**2** 从控制方程入手

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0$$

我们假设 $U(x,t) = \bar{U} + \varepsilon \tilde{U}(x,t)$，其中 $\bar{U}$ 是不随时间和空间变化的常数。

$$\Rightarrow \frac{\partial U}{\partial t} = \frac{\partial}{\partial t} \left[ \bar{U} + \varepsilon \tilde{U} \right] = \varepsilon \frac{\partial \tilde{U}}{\partial t}$$

流量向量也可以展开为：

$$F\left(U\left(x,t\right)\right)=F\left(\bar{U}+\varepsilon\tilde{U}\left(x,t\right)\right)$$
$$\Rightarrow F\left(U\right)\cong F\left(\tilde{U}\right)+\varepsilon\left.\frac{\partial F}{\partial U}\right|_{\bar{U}}\tilde{U}\left(x,t\right)+O\left(\varepsilon^2\right)$$

注意 1： $$\frac{\partial F}{\partial U}=\begin{pmatrix}\frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3}\\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3}\\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3}\end{pmatrix}$$

注意 2：确信你对向量序列的泰勒级数是清楚的。让我们看看它的第一个分量是什么样的

$$F_1\left(U\right)\cong F_1\left(\bar{U}\right)+\varepsilon\left.\frac{\partial F_1}{\partial u_1}\right|_{U=\bar{U}}\tilde{u}_1+\varepsilon\left.\frac{\partial F_1}{\partial u_2}\right|_{U=\bar{U}}\tilde{u}_2+\varepsilon\left.\frac{\partial F_1}{\partial v_3}\right|_{U=\bar{U}}\tilde{u}_3+O\left(\varepsilon^2\right)$$

下面计算$\frac{\partial F}{\partial x}$：

$$\frac{\partial F}{\partial x}\cong\frac{\partial}{\partial x}\left[F\left(\tilde{U}\right)+\varepsilon\left.\frac{\partial F}{\partial U}\right|_{U=\bar{U}}\tilde{U}\left(x,t\right)+O\left(\varepsilon^2\right)\right]$$
$$\cong\varepsilon\left.\frac{\partial F}{\partial U}\right|_{U=\bar{U}}\frac{\partial\tilde{U}}{\partial x}+O\left(\varepsilon^2\right)$$

比较上述结果控制方程有如下形式：

$$\varepsilon\frac{\partial\tilde{U}}{\partial t}+\varepsilon\left.\frac{\partial F}{\partial U}\right|_{U=\bar{U}}\frac{\partial\tilde{U}}{\partial x}+O\left(\varepsilon^2\right)=0$$

然后，取极限$\varepsilon\to 0,\ O\left(\varepsilon^2\right)$这一项很小，因此有：

$$\frac{\partial\tilde{U}}{\partial t}+\left.\frac{\partial F}{\partial U}\right|_{U=\bar{U}}\frac{\partial\tilde{U}}{\partial x}=0$$

**3.** 有两种等价的方法可以用来求$\frac{\partial F}{\partial U}$，首先，回顾偏导数的意义，有

$\dfrac{\partial F_1}{\partial u_2} \equiv$“保持$u_1$和$u_3$不变时，$F_1$关于$u_1$的导数”。

$$\Rightarrow \frac{\partial F_1}{\partial u_2} = \left.\frac{\partial F_1}{\partial u_2}\right|_{u_1,u_3\text{固定}}$$

**方法#1：** 将$F_i$用$u_j$表示

$$\Rightarrow F_1 = \rho u = u_2$$

$$F_2 = \rho u^2 + \rho = \frac{\left(\rho u^2\right)}{\rho} + (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right)$$

$$= \frac{u_2^2}{u_1} + (\gamma - 1)\left(u_3 - \frac{1}{2}\frac{u_2^2}{u_1}\right)$$

$$\Rightarrow F_2 = \left[1 + \frac{1}{2}(\gamma - 1)\right]\frac{u_2^2}{u_1} + (\gamma - 1)u_3$$

$$\boxed{F_2 = \frac{1}{2}(3 - \gamma)\frac{u_2^2}{u_1} + (\gamma - 1)u_3}$$

$$F_3 = \rho u H$$

$$= \rho u\left(E + \frac{P}{\rho}\right)$$

$$= \frac{(\rho u)(\rho E)}{\rho} + \frac{(\rho u)}{\rho}P$$

$$= \frac{u_2 u_3}{u_1} + \frac{u_2}{u_1}\left[(\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right)\right]$$

$$F_3 = \gamma\frac{u_2 u_3}{u_1} + -\frac{1}{2}(\gamma - 1)\frac{u_2^3}{u_1^3}$$

然后，我们可得偏导数。例如

$$\boxed{\begin{aligned}\frac{\partial F_1}{\partial u_1}&=\frac{\partial}{\partial u_1}(u_2)=0\\ \frac{\partial F_1}{\partial u_2}&=\frac{\partial}{\partial u_2}(u_2)=1\\ \frac{\partial F_1}{\partial u_3}&=\frac{\partial}{\partial u_3}(u_2)=0\end{aligned}}$$

$$\frac{\partial F_2}{\partial u_1}=\frac{\partial}{\partial u_1}\left[\frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1}+(\gamma-1)u_3\right]=-\frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1}$$

$$\Rightarrow\boxed{\frac{\partial F_2}{\partial u_1}=-\frac{1}{2}(3-\gamma)\frac{(\rho u)^2}{\rho^2}=-\frac{1}{2}(3-\gamma)u^2}$$

其余导数同理可得。

**方法#2：**

在这个方法里，我们用另外的一些变量，然后强制某个$v_i$保持不变来求出导数。例如，我们根据ρ，$u$和$P$的关系求导.

$$F_1=\rho u\Rightarrow dF_1=\rho du+ud\rho$$

但是，若想求$\frac{\partial F_1}{\partial u_1}$，就须固定$u_1$, $u_2$为常数。由$u_2=\rho u\Rightarrow$

$$du_2=\rho du+ud\rho=0\qquad（因为u_2=\rho u固定）$$

代入$dF_1$，显然有，当$u_1$, $u_2$固定时，$dF_1=0.\Rightarrow\quad\frac{\partial F_1}{\partial u_1}=0$

再以$\frac{\partial F_2}{\partial u_1}$为例：

$$\begin{aligned}&F_2=\rho u^2+P\Rightarrow dF_2=u^2d\rho+2\rho udu+dP\\&u_2=\rho u\Rightarrow du_2=\rho du+ud\rho=0\\&u_3=\rho E=\frac{P}{\gamma-1}+\frac{1}{2}\rho u^2\Rightarrow du_3=\frac{1}{\gamma-1}dP+\frac{1}{2}u^2d\rho+\rho udu=0\end{aligned}$$

将$du_1$=0，和$du_2$=0 代入$dF_2$整理后得到：

$$dF_2 = u^2 d\rho + 2\rho u du + \underbrace{(\gamma-1)\left[-\frac{1}{2}u^2 d\rho - \rho u du\right]}_{\because u_3 = \frac{1}{\gamma-1}dP + \frac{1}{2}u^2 d\rho + \rho u du = 0}$$

$$= \left[1 - \frac{1}{2}(\gamma-1)\right]u^2 d\rho + \left[2-(\gamma-1)\right]\rho u du$$

$$= \frac{1}{2}(3-\gamma)u^2 d\rho + (3-\gamma)\rho u\left[-\frac{u}{\rho}d\rho\right]$$

$$dF_2 = -\frac{1}{2}(3-\gamma)u^2 d\rho$$

但是 $du_1 = d\rho \Rightarrow \frac{\partial F_2}{\partial u_1} = -\frac{1}{2}(3-\gamma)u^2$

第二种方法似乎有点令人困惑，它其实也可以用很机械的办法来实现，下面我们看如何来做：

定义一个新的状态向量，称做 $W$：

$$W = \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}$$

计算 $\frac{\partial F}{\partial W}$ 和 $\frac{\partial U}{\partial W}$. 因为向量 $W$ 的导数很容易求得，我们可得：

$$\frac{\partial F}{\partial U} = \frac{\partial F}{\partial W}\left(\frac{\partial W}{\partial U}\right) = \frac{\partial F}{\partial W}\left(\frac{\partial U}{\partial W}\right)^{-1}$$

**4 .**

将 $\tilde{U}(x-\lambda t)$ 代入控制方程：

$$\frac{\partial \tilde{U}(x-\lambda t)}{\partial t} + \frac{\partial F}{\partial U}\frac{\partial \tilde{U}(x-\lambda t)}{\partial x} = 0$$

$$-\lambda \frac{\partial \tilde{U}}{\partial \eta} + \frac{\partial F}{\partial U}\frac{\partial \tilde{U}}{\partial \eta} = 0 \quad 其中\ \eta = x - \lambda t$$

$$\Rightarrow \left(-\lambda I + \frac{\partial F}{\partial U}\right)\frac{\partial \tilde{U}}{\partial \eta} = 0$$

这只有当λ是$\frac{\partial F}{\partial U}$的特征根时才成立，否则有$\frac{\partial \tilde{U}}{\partial \eta} = 0.$

下面来求$\frac{\partial F}{\partial U}$的特征根。我们先把 $E$ 用 $v$ 和 $c$ 来表示：

$$E = c_v T + \frac{1}{2}u^2 = c_v \frac{\gamma RT}{\gamma R} + \frac{1}{2}u^2 = \frac{c_v}{\gamma R}c^2 + \frac{1}{2}u^2$$

$$\boxed{E = \frac{1}{\gamma(\gamma-1)}c^2 + \frac{1}{2}u^2}$$

$$\frac{\partial F}{\partial U} - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u - \lambda & \gamma - 1 \\ (\gamma-1)u^3 - \gamma u E & -\frac{3}{2}(\gamma-1)u^2 + \gamma E & \gamma u - \lambda \end{pmatrix}$$

$$\det\left(\frac{\partial F}{\partial U} - \lambda I\right) = -\lambda\left\{\left[(3-\gamma)u - \lambda\right](\gamma u - \lambda) - (\gamma-1)\left[-\frac{3}{2}(\gamma-1)u^2 + \gamma E\right]\right\}$$
$$-\left\{\frac{1}{2}(\gamma-3)u^2(\gamma u - \lambda) - (\gamma-1)\left[(\gamma-1)u^3 - \gamma u E\right]\right\}$$

按λ的幂整理：

$$-\lambda^3 + \left\{\gamma u + (3-\gamma)u\right\}\lambda^2$$
$$+\left\{-\gamma(3-\gamma)u^2 + (\gamma-1)\left[-\frac{3}{2}(\gamma-1)u^2 + \gamma E\right] + \frac{1}{2}(\gamma-3)u^2\right\}\lambda$$
$$+\frac{1}{2}(3-\gamma)\gamma u^3 + (\gamma-1)\left[(\gamma-1)u^3 - \gamma u E\right]$$
$$= 0$$

简化得：

$$
\begin{aligned}
&-\lambda^3+3u\lambda^2\\
&+\left\{\left[-3\gamma+\gamma^2-\frac{3}{2}\left(\gamma^2-2\gamma+1\right)+\frac{1}{2}(\gamma-3)\right]u^2+\gamma\left(\gamma-1\right)E\right\}\lambda\\
&+\left[\frac{1}{2}(3-\gamma)\gamma+\gamma^2-2\gamma+1\right]u^3-\gamma\left(\gamma-1\right)uE\\
&=0
\end{aligned}
$$

$$
\begin{aligned}
&-\lambda^3+3u\lambda^2\\
&+\left\{\left[-\frac{1}{2}\gamma^2+\frac{1}{2}\gamma-3\right]u^2+\gamma\left(\gamma-1\right)E\right\}\lambda\\
&+\left[\frac{1}{2}\gamma^2-\frac{1}{2}\gamma+1\right]u^3-\gamma\left(\gamma-1\right)uE\\
&=0
\end{aligned}
$$

代入 $E$ 的表达式：

$$
\begin{aligned}
&-\lambda^3+3u\lambda^2+\left\{\left[-\frac{1}{2}\gamma^2+\frac{1}{2}\gamma+3\right]u^2+c^2+\frac{1}{2}\gamma\left(\gamma-1\right)u^2\right\}\lambda\\
&+\left[\frac{1}{2}\gamma^2-\frac{1}{2}\gamma+1\right]u^3-uc^2-\frac{1}{2}\gamma\left(\gamma-1\right)u^3\\
&=0
\end{aligned}
$$

$$-\lambda^3+3u\lambda^2+\left(-3u^2+c^2\right)\lambda+u^3-uc^2=0$$

提出因子λ-$u$，

$$\left(\lambda-u\right)\left(-\lambda^2+2u\lambda-u^2+c^2\right)=0$$

$$\left(\lambda-u\right)\left(-\lambda+u-c\right)\left(\lambda-u-c\right)=0$$

$$\boxed{\Rightarrow\lambda=u,\ \ u-c,\ \ u+c}$$

## 问题#2：逆风差分和数值损耗

**1.**

$$
\begin{aligned}
u_i\frac{\partial T}{\partial x_i}&=u_i\delta_{2x}T_i-\frac{1}{2}\left|u_i\right|\Delta x\delta_x^2T_i\\
&=u_i\frac{T_{i+1}-T_i}{2\Delta x}-\frac{1}{2}\left|u_i\right|\frac{T_{i+1}-2T_i+T_{i-1}}{2\Delta x}
\end{aligned}
$$

$$u_i \frac{\partial T}{\partial x_i} = u_i \frac{T_i - T_{i-1}}{\Delta x} = u_i \delta_x^- T_i$$

现令$u_i$>0:

$$\Rightarrow u_i \frac{\partial T}{\partial x_i} = u_i \frac{T_{i+1} - T_i}{2\Delta x} - \frac{1}{2} u_i \frac{T_{i+1} - 2T_i + T_{i-1}}{2\Delta x}$$

$$\boxed{u_i \frac{\partial T}{\partial x_i} = u_i \frac{T_i - T_{i-1}}{\Delta x} = u_i \delta_x^- T_i}$$

现令$u_i < c$:

$$u_i \frac{\partial T}{\partial x_i} = u_i \frac{T_{i+1} - T_i}{2\Delta x} + \frac{1}{2} u_i \frac{T_{i+1} - 2T_i + T_{i-1}}{2\Delta x}$$

$$\boxed{u_i \frac{\partial T}{\partial x_i} = u_i \frac{T_{i+1} - T_i}{\Delta x} = u_i \delta_x^+ T_i}$$

## 2.

由（9）式，乘以第二个导数近似的那个系数是$\frac{1}{2}|u_i|\Delta x$.

$$\boxed{\Rightarrow K_{num} = \frac{1}{2}|u_i|\Delta x}$$

## 3. 问题求对于$K = K_{num}$，需要多大的$\Delta x$.

$$K = \frac{1}{2}|u_i|\Delta x$$

$$\Rightarrow \Delta x = \frac{2K}{|u_i|}$$

$$\boxed{\frac{\Delta x}{c} = \frac{2K}{|u_i|c} = \frac{2}{\mathrm{Re}}}$$

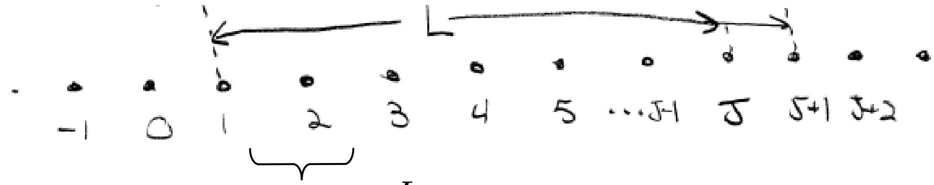
所以，如果雷诺数$\boxed{\mathrm{Re} = 10^7 \Rightarrow \frac{\Delta x}{c} = 2 \times 10^{-4}}$

这暗示着：

$$\boxed{\begin{array}{l} N_x = x\text{方向的分割数} = \frac{c}{\Delta x} = \frac{\mathrm{Re}}{2} = 5 \times 10^3 \\ \mathrm{Re}\text{太大了，典型}\frac{\Delta x}{c} \approx 0.01 \end{array}}$$

# 问题#3：对流方程的矩阵稳定性分析

**1.** $x$ 从 $0 \rightarrow L$ 共有 $J$ 个格点：




因此，
$$\Delta x = \frac{L}{J-1}$$

注意：周期性要求$T_{J+1}=T_1$, $T_{J+2}=T_2$, 等等。考虑节点 3：

$$\frac{\partial T_3}{\partial t} + u\delta_{2x}T_3 = 0 \Rightarrow$$
$$\frac{\partial T_3}{\partial t} + u\frac{T_4 - T_2}{2\Delta x} = 0$$

除了周期边界的处理之外，这对于其它所有内节点都是同样的。

对于节点 1 和 $J$, 需要有些小变化。

$$\frac{\partial T_1}{\partial t} + u\frac{T_2 - T_0}{2\Delta x} = 0, \quad 但 T_0 = T_J$$

$$\boxed{\Rightarrow \frac{\partial T_1}{\partial t} + u\frac{T_2 - T_J}{2\Delta x} = 0}$$

类似地，

$$\frac{\partial T_J}{\partial t} + u\frac{T_{J+1} - T_{J-1}}{2\Delta x} = 0 \Rightarrow \frac{\partial T_J}{\partial t} + u\frac{T_1 - T_{J-1}}{2\Delta x} = 0$$

注意：我们不需要$T_{J+1}$的式子，因为$T_{J+1}=T_1$，因此，它与$T_1$的控制方程是等价的。故我们可写出：

$$\frac{\partial}{\partial t}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{J-1} \\ T_J \end{pmatrix} +$$

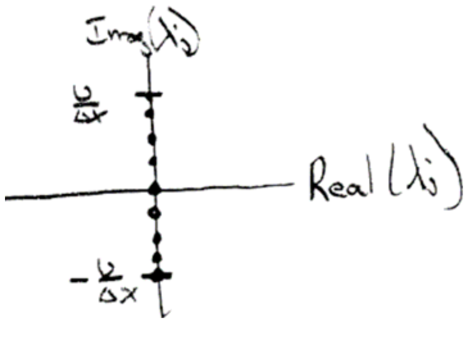
$$\begin{pmatrix} 0 & \frac{u}{2\Delta x} & 0 & \cdots & \cdots & \cdots & -\frac{u}{2\Delta x} \\ -\frac{u}{2\Delta x} & 0 & \frac{u}{2\Delta x} & 0 & \cdots & \cdots & 0 \\ 0 & -\frac{u}{2\Delta x} & 0 & \frac{u}{2\Delta x} & 0 & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & 0 \\ 0 & 0 & 0 & \cdots & -\frac{u}{2\Delta x} & 0 & \frac{u}{2\Delta x} \\ \frac{u}{2\Delta x} & 0 & 0 & 0 & \cdots & -\frac{u}{2\Delta x} & 0 \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{J-1} \\ T_J \end{pmatrix} = 0$$

$$\boxed{\Rightarrow a = \frac{-u}{2\Delta x} \qquad b = 0 \qquad c = \frac{u}{2\Delta x}}$$

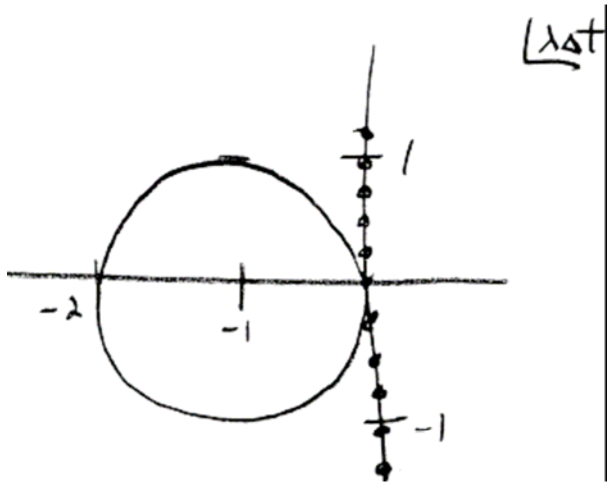
**2.** 因为 $a+b=0$，且 $b=0$

$$\Rightarrow \lambda_j = i\frac{u}{\Delta x}\sin\left(\frac{2\pi j}{J}\right) \qquad 对于 j = 0 \rightarrow J-1$$

所以特征根是纯虚数，可以扩展到 $\pm i\frac{v}{\Delta x}$




**3. 向前欧拉方程的稳定区域是：**




显然没有哪个模式位于向前欧拉方程的稳定区域内，因此，

$$\boxed{\Delta t = 0!}$$

## 4．　4 阶龙格库塔格式的稳定区域

4 阶龙格库塔的稳定区域可以用程序 **rkstab.m** 画出。

对于 4 阶龙格库塔，$g$ 的表达式用

$$g = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4$$

$$|\lambda \Delta t| \le 2.7$$

$$\frac{|v|\Delta t}{\Delta x} \le 2.7$$

或者 $\Delta t \le 2.7\dfrac{\Delta x}{|v|}$