14.126 Game Theory

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Lecture 8:

Repeated Games with Imperfect Monitoring

(APS 1990, continued)

Signalling Games

Alternating Offers Bargaining

Proof of Self-Generation (continued): $v(\sigma(w)) = w$

Let $x \in B(W)$. Then:

$$\begin{aligned} v(\sigma(x)) &= (1-\delta)\Pi(a(x)) + \delta \sum_{p\in\Omega} f(p\,|\,a) v\left(\sigma\left(U(x)(p)\right)\right) \\ x &= (1-\delta)\Pi(a(x)) + \delta \sum_{p\in\Omega} f(p\,|\,a) U(x)(p) \end{aligned}$$

implying:

$$v(\sigma(x)) - x = \delta \sum_{p\in\Omega} f(p\,|\,a)\left[v\left(\sigma\left(U(x)(p)\right)\right) - U(x)(p)\right]$$

$$\Rightarrow |v(\sigma(x)) - x| \le \delta \sup_{y\in W} |v(\sigma(y)) - y| \le \delta \sup_{y\in B(W)} |v(\sigma(y)) - y|$$

Taking sup on the l.h.s.:

$$\sup_{x\in B(W)} |v(\sigma(x)) - x| \le \delta \sup_{y\in B(W)} |v(\sigma(y)) - y|$$

Since the supremum is finite $v(\sigma(x)) = x$ for any $x \in B(W)$.

Proof of Self-Generation (continued): $\sigma(w)$ is a SE.

To verify that there there are no profitable single deviations, take any history p^t of public signals. Let $x = U^t(w)(p^t) \in W$. Since $(a(x), U(x))$ is admissible w.r.t. W:

$$(1-\delta)\Pi_i(a_i(x), a_{-i}(x)) + \delta \sum_{p\in\Omega} f(p \,|\, a(x))U_i(x)(p)$$

$$\geq (1-\delta)\Pi_i(a_i', a_{-i}(x)) + \delta \sum_{p\in\Omega} f(p \,|\, a_i', a_{-i}(x))U_i(x)(p)$$

Plug in

$a(x) = \sigma(w)(t+1)(p^t)$,

$\sigma(w)|_{p^t,p} = \sigma\left(U^{t+1}(w)(p^t, p)\right) = \sigma\left(U(x)(p)\right)$, and

$v\left(\sigma\left(U(x)(p)\right)\right) = U(x)(p)$.

Hence i has no profitable deviations after p^t. $\square$

Proof of Factorization:

$$V = B(V)$$

By Self-Generation, we only need to show $V \subset B(V)$. Let $w = v(\sigma) \in V$ where σ is a SE. We need to find a pair (a, u) admissible w.r.t V s.t. $w = E(a, u)$.

Set $a := \sigma(1)$ and $u(p) := v(\sigma|_{p,a})$ for any $p \in \Omega$. Then $w = v(\sigma) = E(a, u)$

$\sigma|_{p,a}$ is an SE (Why?). Hence $u(p) \in V$.

Consider a deviation by player i where he plays a'_i at $t = 1$ and plays according to σ_i subsequently. Since σ is a SE, this is not profitable, i.e.:

$$(1-\delta)\Pi_i(a'_i, a_{-i}) + \delta \sum_{p \in \Omega} f(p \,|\, a'_i, a_{-i}) v_i\,(\sigma|_{p,a})$$

$$\leq (1-\delta)\Pi_i(a) + \delta \sum_{p \in \Omega} f(p \,|\, a) v_i\,(\sigma|_{p,a})$$

Therefore $E_i(a'_i, a_{-i}, u) \leq E_i(a, u)$. $\square$

Proof of Monotonicity

Fact: $B^2\left(V^1\right) = B^2\left(co\left(V^1\right)\right)$. (see the last slide)

Let $\delta_2 > \delta_1$. By Thm 1, we only need to show V^1 is a self-generating set at δ_2. Let $w \in V^1$. Then at δ^1, there is an admissible pair (a, u^1) w.r.t V^1 s.t. $w = E^1(a, u^1)$, i.e.:

$$w = (1 - \delta_1)\Pi(a) + \delta_1 \sum_{p \in \Omega} f(p \,|\, a) u^1(p)$$

$$\begin{aligned} \Leftrightarrow w - \Pi(a) &= \frac{\delta_1}{1-\delta_1} \sum_{p \in \Omega} f(p \,|\, a) \left[u^1(p) - w\right] \\ &= \frac{\delta_2}{1-\delta_2} \sum_{p \in \Omega} f(p \,|\, a) \kappa \left[u^1(p) - w\right]. \end{aligned}$$

where $\kappa := [\delta_1/(1-\delta_1)]/[\delta_2/(1-\delta_2)] \in (0, 1)$.

$u^2(p) - w := \kappa[u^1(p) - w] \Rightarrow u^2(p) = \kappa u^1(p) + (1-\kappa)w \in co(V^1)$.

It remains to verify incentives for (a, u^2) at δ_2.

Proof of Monotonicity (continued)
Verifying Incentives for (a, u^2) at δ_2

Incentives for (a, u^1) at δ_1:

$$\Pi_i(a_i', a_{-i}) - \Pi_i(a) \;\leq\; \frac{\delta_1}{1-\delta_1} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i})\Big]\, u_i^1(p)$$

This is equivalent to:

$$\Pi_i(a_i', a_{-i}) - \Pi_i(a) \;\leq\; \frac{\delta_1}{1-\delta_1} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i})\Big] \Big[u_i^1(p) - w_i\Big]$$

$$\begin{aligned} \text{r.h.s} \;&=\; \frac{\delta_2}{1-\delta_2} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i})\Big]\, \kappa \Big[u_i^1(p) - w_i\Big] \\ &=\; \frac{\delta_2}{1-\delta_2} \sum_{p \in \Omega} \Big[f(p \,|\, a) - f(p \,|\, a_i', a_{-i})\Big] \Big[u_i^2(p) - w_i\Big] \end{aligned}$$

Hence the incentives for (a, u^2) at δ_2 are satisfied. $\square$

Technical Results

Theorem 3 *(Sufficiency of Bang-Bang Reward Functions) Let W be compact. Then for any (a,u) that is admissible w.r.t. $co(W)$, there is u' s.t. (i) (a,u') is admissible w.r.t. $ext(W)$ and (ii) $E(a,u') = E(a,u)$.*

Corollary For any compact W, $B(W) = B\left(co(W)\right)$.

Theorem 4 *V is compact.*

Theorem 5 *(Algorithm) Let W be a compact set such that $V \subset B(W) \subset W$. Then $B^k(W) \searrow V$.*

"Theorem 7" *(Necessity of Bang-Bang Reward Functions) Under certain conditions, the reward functions faced by players in efficient equilibria* **must take** *values in $ext(V)$.*

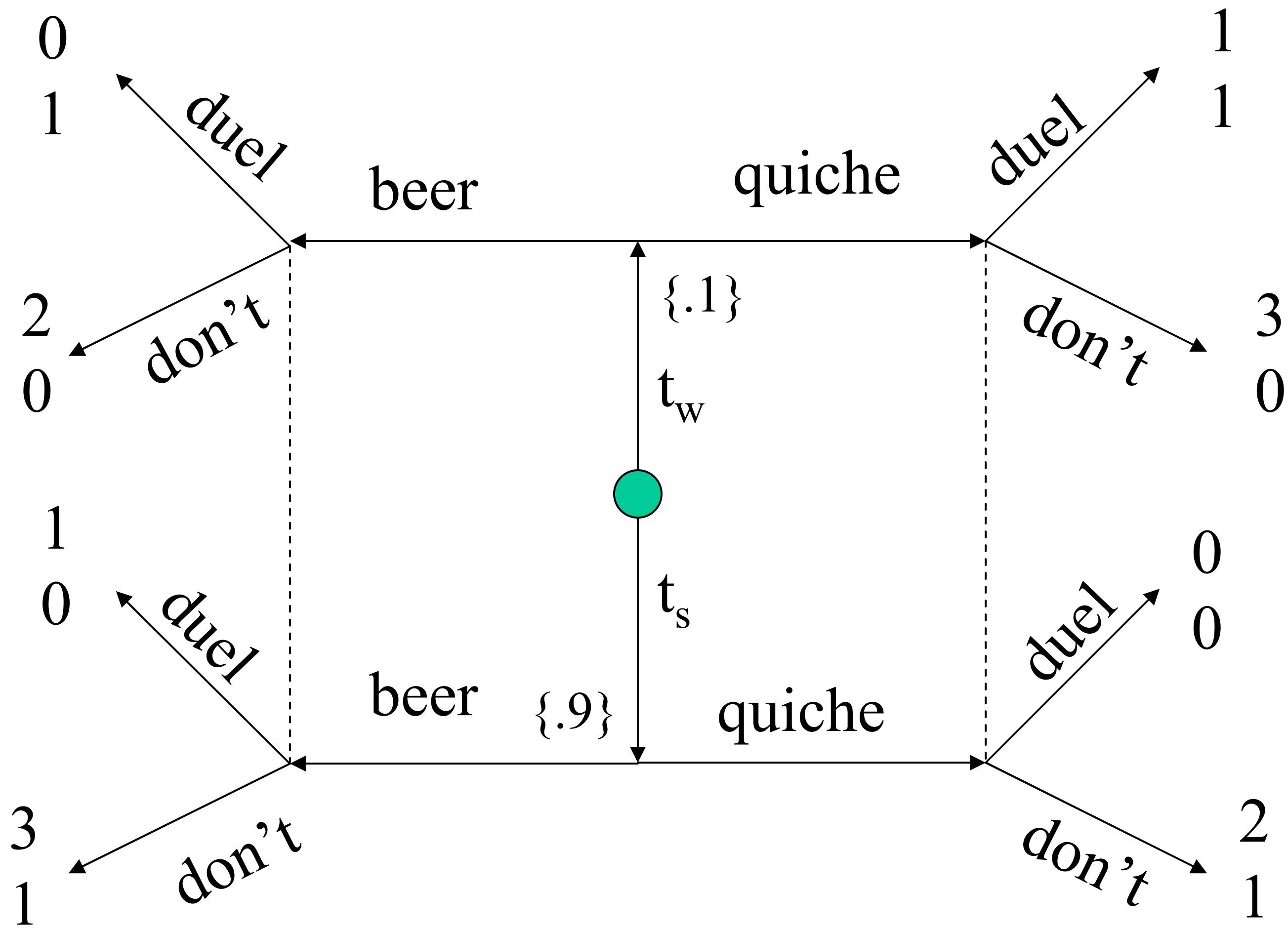
Signaling Games
& the Intuitive Criterion
(Cho & Kreps, 1987)

Signaling Game -- Definition

- Two Players: (S)ender, (R)eceiver

1. Nature selects a type t from T with probability p(t);
2. Sender observes t, and then chooses a message m from M;
3. Receiver observes m (but not t), and then chooses an action a from A;
4. Payoffs are $U_S(t,m,a)$ and $U_R(t,m,a)$.

Beer – Quiche


0
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duel
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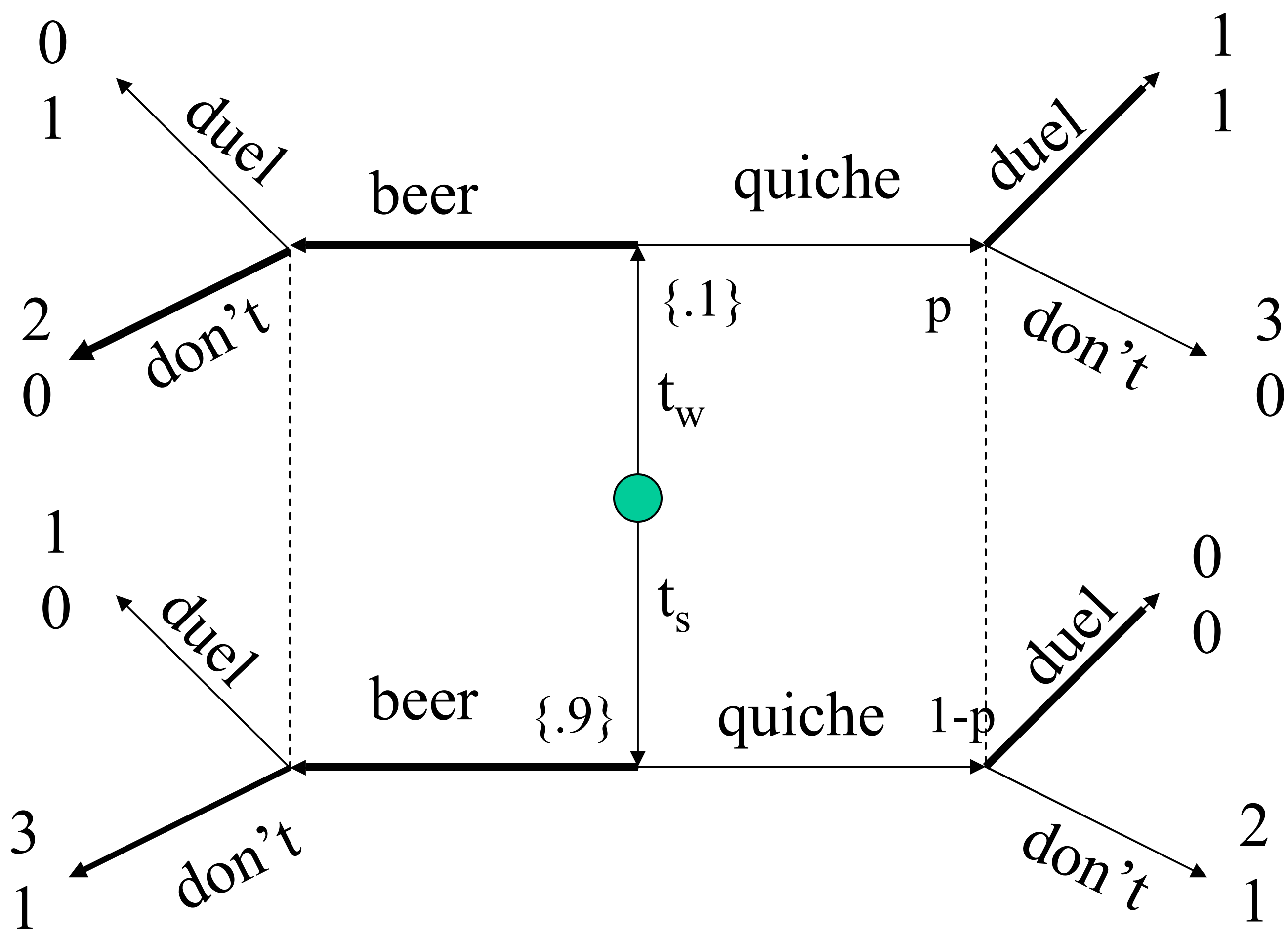
1

Types of Sequential Equilibria (SE)

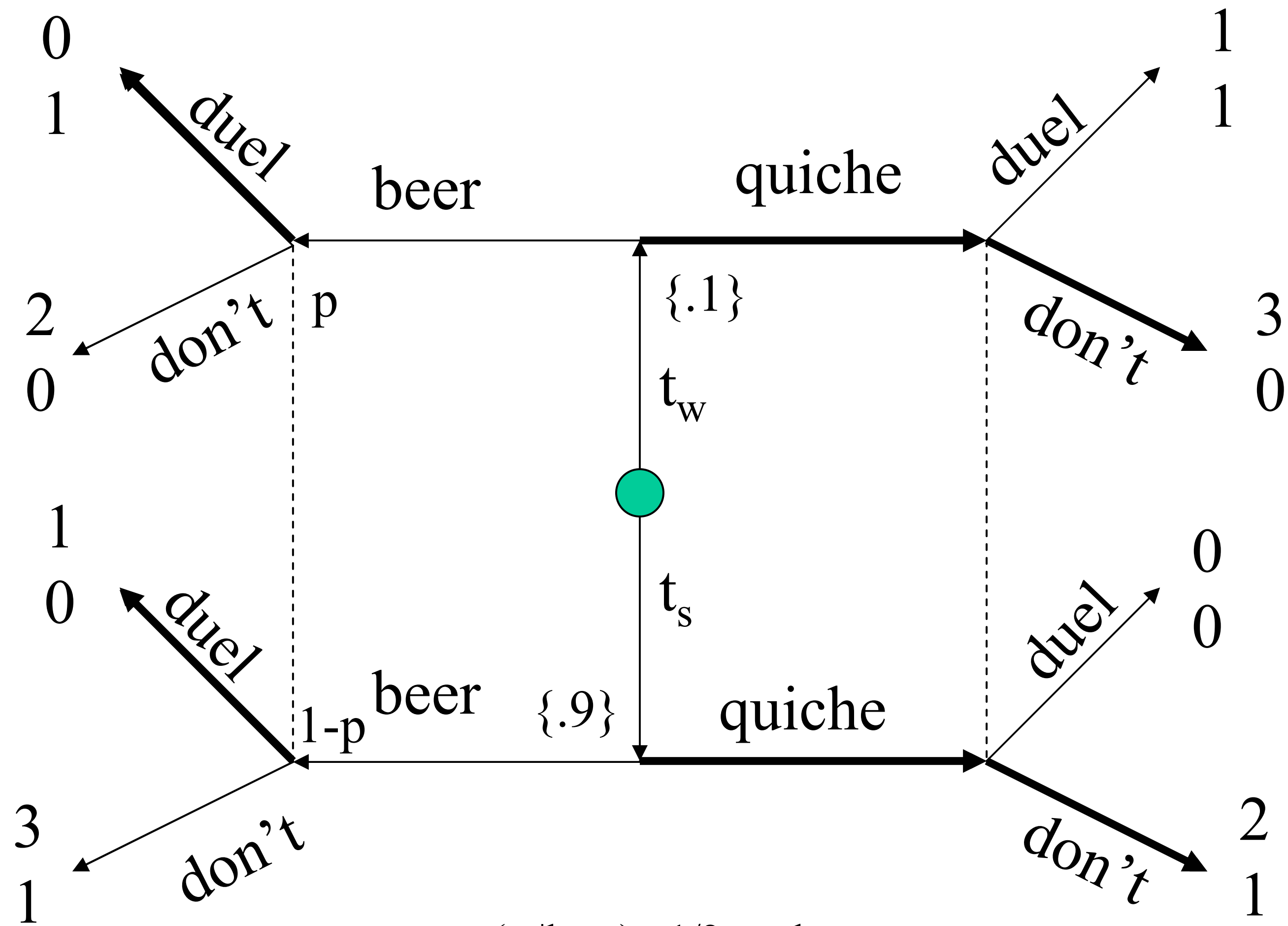
- A **pooling equilibrium** is an equilibrium in which all types of sender send the same message.
- A **separating equilibrium** is an equilibrium in which all types of sender send different messages.
- A **partially separating/pooling equilibrium** is an equilibrium in which some types of sender send the same message, while some others sends some other messages.

Pooling SE where "Beer" is Played



p=$\mu(t_w|\text{quiche}) > 1/2$, and
p=1/2 & Prob (duel | quiche) ≥1/2

Pooling SE where "Quiche" is Played



p=μ(t_w|beer) >1/2, and

p=1/2 & Prob (duel | beer) ≥1/2

Intuitive Criterion

Given a SE, if t's equilibrium payoff denoted $U^*(t)$ is greater than t's highest possible payoff from m, i.e:

$$U^*(t) > \max_{a \in A} U_S(t, m, a) \qquad (*)$$

then after receiving message m_j, the Receiver's should place zero probability on Sender being of type t, i.e. $\mu(t|m)=0$.

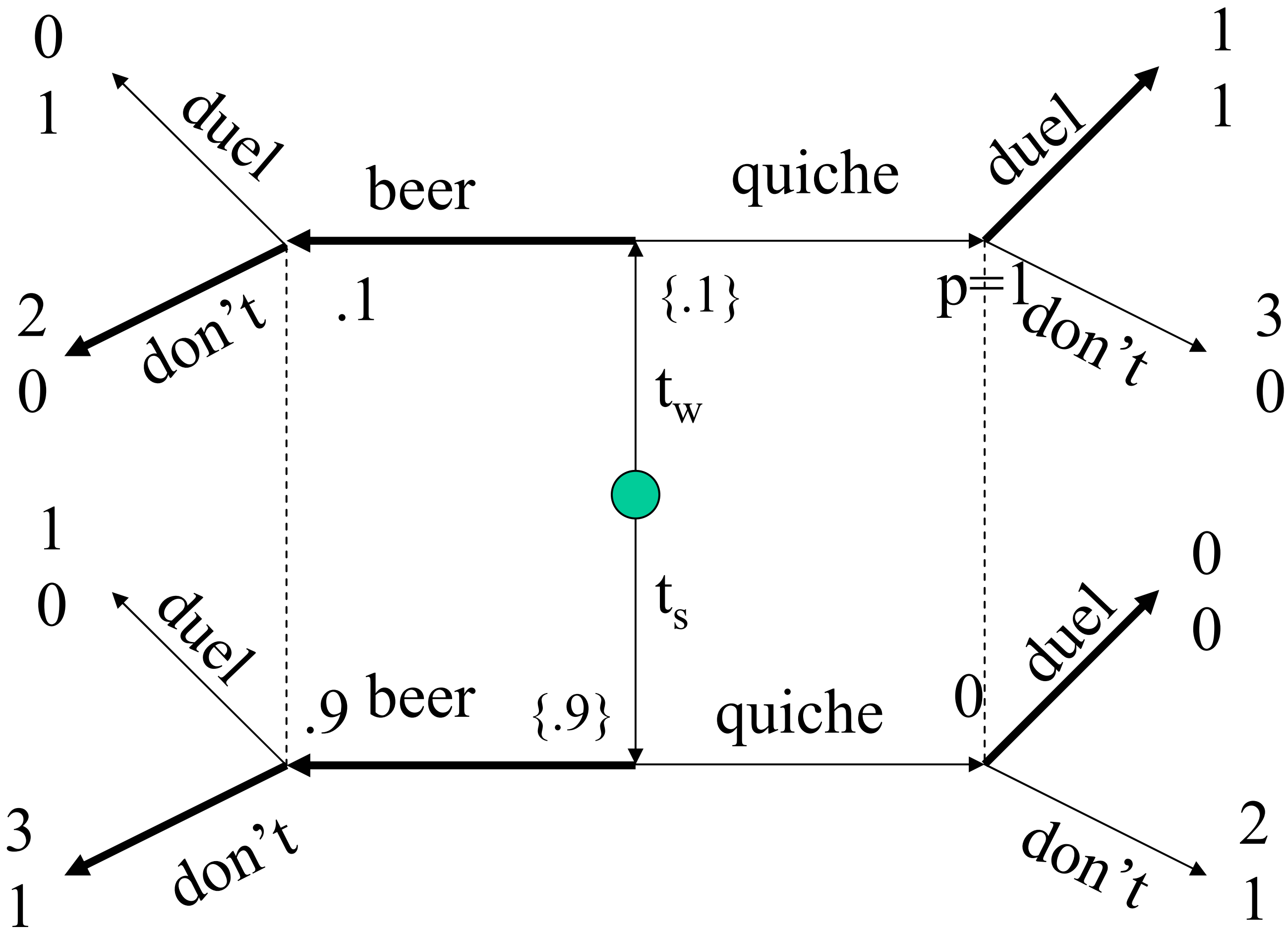
Note: This is possible if (*) does not hold for at least one type in T.

Cho & Kreps:

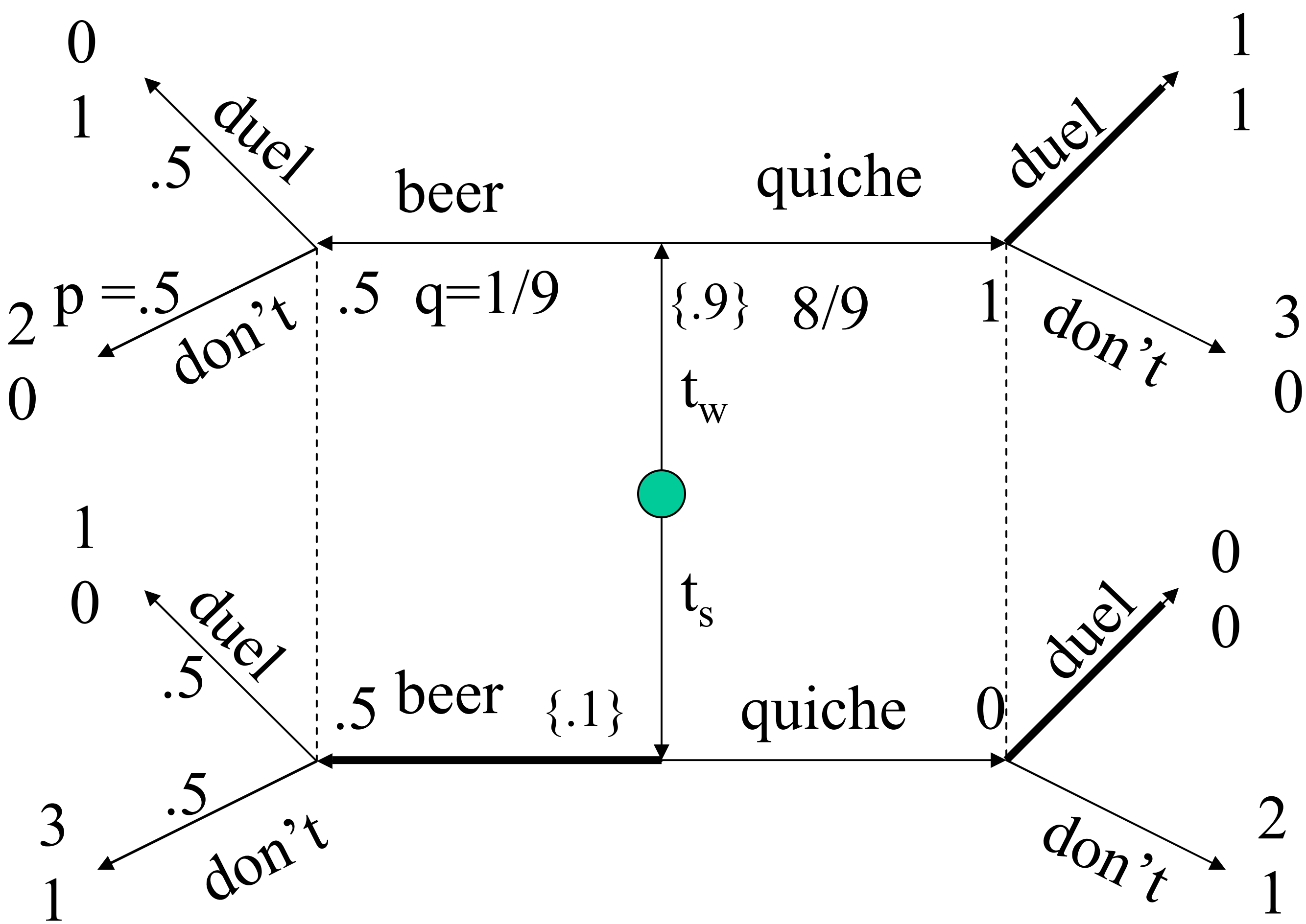
It is as if the sender, if he were of strong type, is (by drinking beer) implicitly making the speech:

“*I am drinking beer, which ought to convince you that I am of the strong type. For (given the Quiche-pooling equilibrium) I would never wish to drink beer if I were of the weak type. While, if I am of the strong type, and if drinking beer convinces you, then as you see it is in my interest to drink beer.*”

The only pooling SE that survives the Intuitive Criterion



A Mixed SE



0
1
duel
.5
beer
quiche
duel
1
1
p =.5
2
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don't
.5
q=1/9
{.9}
8/9
1
don't
3
0
t_w
t_s
1
0
duel
.5
.5
beer
{.1}
quiche
0
duel
0
0
.5
3
1
don't
don't
2
1

Alternating Offers Bargaining
Infinite Horizon
(Rubinstein, 1982)

The Model

- Two players $N = \{1,2\}$ bargain over sharing a dollar over time.
- A division where player 1 receives x and player 2 receives y, is denoted by (x,y) where $x+y=1$, $x \geq 0$, $y \geq 0$.
- Both players are impatient. They have a discount factor $\delta \in (0,1)$. If they agree at time t about a division (x_t, y_t), then the utility of each player is:

$$U_1(x_t,t) = \delta^{t-1} x_t \quad \text{and} \quad U_2(y_t,t) = \delta^{t-1} y_t$$

- If they never reach an agreement, then they both receive 0.

Alternating offers bargaining model infinite time horizon

$T = \{1,2,\ldots\}$

If t is odd,

- Player 1 offers some (x_t,y_t),
- Player 2 Accepts or Rejects the offer
- If the offer is Accepted, the game ends yielding $(\delta^{t-1}x_t, \delta^{t-1}y_t)$,
- Otherwise, we proceed to date $t+1$.

If t is even

- Player 2 offers some (x_t,y_t),
- Player 1 Accepts or Rejects the offer
- If the offer is Accepted, the game ends yielding payoff $(\delta^{t-1}x_t, \delta^{t-1}y_t)$,
- Otherwise, we proceed to date $t+1$.

SPE of ∞-period bargaining

Theorem: At any t, proposer offers the other player $\delta/(1+\delta)$, keeping himself $1/(1+\delta)$, while the other player accepts an offer iff she gets at least $\delta/(1+\delta)$.

Proof: Single-deviation principle: Take any t, at which i offers, j accepts/rejects. According to the strategies in the continuation game, at t+1, j will get $1/(1+\delta)$. Hence, it is a best response for j to accept an offer iff she gets at least $\delta/(1+\delta)$. Given this, i must offer $\delta/(1+\delta)$.

Uniqueness of SPE

Uniqueness of SPE payoffs:

M=supremum SPE payoff of a player in a subgame where she starts making an offer.

m=infimum SPE payoff of a player in a subgame where she starts making an offer.