Learning
Replicator dynamics &
Adjustment with persistent noise

14.126 Game Theory

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Road Map

1. Definition
2. Asymptotic behavior of Replicator dynamics
    1. RD vs. Rationalizability
    2. RD vs. ESS
    3. RD vs. Perfect equilibria
3. Generalization of RD
4. Adjustment models with persistent randomness

Notation

- G = (S,A) a symmetric, 2-player game where
  - S is the strategy space;
  - $A_{i,j} = u_1(s_i,s_j) = u_1(s_i,s_j)$.
- x,y are mixed strategies; $u(x,y) = x^T Ay$. u(s,y).
- ax+(1-a)y.
- u(ax+(1-a)y,z) = au(x,z) +(1-a)u(y,z)

Replicator dynamics

- $p_i(t)$ = #people who plays s_i at t;
- p(t) = total population at t.
- $x_i(t) = p_i(t)/p(t)$; $x(t) = (x_1(t),\ldots, x_k(t))$.
- $\dot{x}_i = [u(s_i, x) - u(x,x)]x_i \equiv u(s_i - x, x)x_i$

Rationalizability

- Let x(.) be the solution to replicator dynamics starting at x_0.

Theorem: If a pure strategy i is strictly dominated by y and x_0 is in the interior, then $\lim_t x_i(t) = 0$.

Proof: Define $v_i(x) = \log(x_i) - \Sigma_j y_j \log(x_j)$. Then,

$$\frac{dv_i(x(t))}{dt} = \frac{\dot{x}_i}{x_i} - \sum_j y_j \frac{\dot{x}_j}{x_j} = u(s_i - x, x) - \sum_j y_j u(s_j - x, x) = u(s_i - y, x).$$

Hence, $v_i(x(t)) \to -\infty$., i.e., $x_i(t) \to 0$.

Theorem: If i is not rationalizable and x_0 is in the interior, then limt $x_i(t,x_0) = 0$.

Theorem

Every ESS x is an asymptotically stable steady state of replicator dynamics. (If the individuals can inherit the mixed strategies, the converse is also true.)

Proof: Define C = supp(x), Q = {y|C $\subset$ supp(y)}, $H(y) = \Sigma_{i\in C} x_i \log(y_i)$.

1. x is a local maximum of H, and
2. $\exists$a neighborhood n(x) s.t. H is increasing along any trajectory in Q$\cap$n(x).

$$\dot{H} = \sum_{i\in C} x_i \frac{\dot{y}_i}{y_i} = \sum_{i\in C} x_i u(s_i - y, y) = u(x - y, y) > 0.$$

Theorem

If x is an asymptotically stable steady state of replicator dynamics, then (x,x) is a perfect Nash equilibrium.

<u>Proof:</u>

1. (x,x) is a Nash equilibrium.
   1. x is stable => $dx/dt = u(s_i\text{-}x,x)x_i=0$.
   2. Suppose $(x,x)\notin NE$.
   3. $\exists i \notin supp(x)$: $u(s_i\text{-}x,x) > 0$. [by 1 and 2]
   4. $\exists\delta>0, n(x)$: $u(s_i\text{-}y,y) > \delta \ \forall y \in n(x)$.
   5. $\xi_i(t,y^0) > y^0{}_i e^{\delta t}$ if $\xi_i(.,y^0)$ remained in n(x).
2. x is not weakly dominated.

General dynamics

Definition: A process is *payoff monotone* iff, at each interior x,

$$u(s_i,x) > u(s_j,x) \Leftrightarrow \frac{\dot{x}_i}{x_i} > \frac{\dot{x}_j}{x_j}.$$

Theorem: Under any any "regular," payoff monotone dynamics, if strategy i is eliminated by the process of iterated pure strategy strict dominance, then $\lim_t x_i(t) = 0$.

Adjustment models with Persistent Randomness

Main idea

- There will always be small but positive probability of mutation.
- Then, some of the strict Nash equilibria will **not** be “stochastically stable.”

Stochastic Adjustment

1. Consider a game.
2. Specify a state space Θ, e.g., the # of players playing a strategy.

1.

	A	B
A	2,2	0,0
B	0,0	1,1

2. $\Theta = \{AA, AB, BA, BB\}$

Stochastic Adjustment, continued

3. Specify an adjustment dynamics, e.g., best-response dynamics, with a transition matrix P, where

   $P_{\theta,\xi} = \Pr(\theta \text{ at } t+1 | \xi \text{ at } t)$

ϕ = a probability distribution, a column vector.

3. P =

AA	AB	BA	BB	
1	0	0	0	AA
0	0	1	0	AB
0	1	0	0	BA
0	0	0	1	BB

Stochastic Adjustment, continued

4. Introduce a small noise: Consider P^{ε}, continuous in ε and $P^{\varepsilon} \to P$ as $\varepsilon \to 0$.

   Make sure that there exist a unique ϕ_{ε}^* s.t.

   $\phi_{\varepsilon}^* = P^{\varepsilon} \phi_{\varepsilon}^*$.

4\. $P^{\varepsilon} =$

	AA	AB	BA	BB
	$(1-\varepsilon)^2$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	ε^2
	$(1-\varepsilon)\varepsilon$	ε^2	$(1-\varepsilon)^2$	$(1-\varepsilon)\varepsilon$
	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)^2$	ε^2	$(1-\varepsilon)\varepsilon$
	ε^2	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)\varepsilon$	$(1-\varepsilon)^2$

$\phi_{\varepsilon}^* = (1/4,1/4,1/4,1/4)^T$.

Stochastic Adjustment, continued

5. Verify that $\lim_{\varepsilon\to 0}\phi_{\varepsilon}^* = \phi^*$ exists; compute ϕ^*. (By continuity $\phi^* = P\phi^*$.)
6. Check that ϕ^* is a point mass, i.e.,

$$\phi^*(\theta^*) = 1$$

   for some θ^*.

The strategy profile at θ^* is called *stochastically stable equilibrium*.