# In-Class Problems — Week 3, Fri

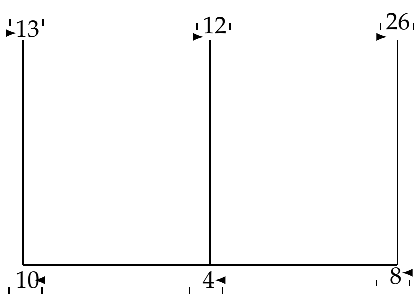
*Definition:* The *composition* of relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is the relation

$S \circ R = \{(a, c) \mid \exists b$ such that $(a, b) \in R \wedge (b, c) \in S\}$.

**Problem 1.** Recently MIT students have been taking a hard look at the haphazard building layout, and have been asking some hard questions. As always they know they can use their superior mathematical skills to get some real answers to those hard questions.

They decide to express the MIT building layout as a relation. Let $C$ be the set of all building numbers and let $R$ be the relation on the set $C$ such that $(a, b) \in R$ if building $a$ and building $b$ are physically adjacent and there is a door between $a$ and $b$ (more importantly, one doesn't have to go outside to get from $a$ to $b$). Note that if $(a, b) \in R$, then $(b, a)$ is also in $R$, so $R$ is a symmetric relation. For convenience, they also define a building to be related to itself, so $(a, a) \in R$.

**(a)** For this part only, let $C$ be the set of MIT buildings 10,13,12,4,8,26. Then $R$ looks like this:




Compute $R^2 = R \circ R$.
Compute $R^3 = R \circ R^2$.

**(b)** Let $R$ be the map for all of MIT. What does the relation $R^2$ represent in terms of connectivity between numbered buildings? Can you generalize this?

**(c)** One of the important questions for course 6 students was, is it possible to get from building 36 to building 10 without crossing more than 5 other buildings? Write a proposition in terms of $R$, using relational and set operators, which is true if this condition is satisfied and false otherwise.

**(d)** The MIT students would like to be able to get from any building to any other building, without having to go outside. Write the condition on $R$ that must be satisfied in order for this to be true.

**(e)** MIT administration, however, wants to keep the number of connections between building as small as possible. In other words, MIT wants the size, $|R|$, of $R$ to be as small as possible. What is the smallest $R$ that satisfies the requirement in part (d)? Is the smallest $R$ unique?

**Problem 2.** A relation $R$ from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$ can be represented as a boolean matrix $M_R$, where $M_R(i,j) = 1$ if the pair $(a_i, b_j) \in R$ and $M_R(i,j) = 0$ if the pair $(a_i, b_j)$ is not in $R$.

We define boolean matrix multiplication to be the same as regular matrix multiplication except that "+" is replaced by $\vee$ (Boolean OR) and "$\times$" is replaced by $\wedge$ (Boolean AND).

**(a)** We have a student set $stud = \{Adrian, Min, Josh\}$, a class set $class = \{6.042, 6.046\}$ and a lecture set $lect = \{Albert, Charles, Radhi\}$. The relation $K$ "is taking class" as a subset of $stud \times class$ is defined by the list: $\{(Adrian, 6.042), (Min, 6.046), (Josh, 6.042), (Josh, 6.046)\}$ and the relation $L$ "is lectured by"as a subset of $class \times lect$ is defined by the list: $\{(6.042, Albert), (6.042, rahdi),$ $(6.046, Charles)\}$. The relation $T$ "is taught by" is the composition of relations $K$ and $L$. Represent relation $T$ in boolean matrix and compare it with the boolean matrix multiplication of relation $K$ and relation $L$.

**(b)** Let $M_P$ be the boolean multiplication of $M_R$ and $M_S$, where $R \subseteq A \times B$ and $S \subseteq B \times C$. Write down the formula of the boolean matrix multiplication of $M_P(i,j)$ in terms of $M_R$ and $M_S$. [1]

**(c)** Prove that boolean multiplication of $M_R$ and $M_S$ is equal to $M_{S \circ R}$.

**(d)** What does the regular multiplication of $M_R$ and $M_S$ give you?

**Problem 3.** The term *six degrees of separation* implies that everyone knows everyone else indirectly through at most 6 other people. Discuss how you would write a computer program to determine if six degrees of separation holds within our 6.042 class.

[1] Recall that the *composition* of relations $R \subseteq A \times B$ and $S \subseteq B \times C$ is the relation $S \circ R = \{(a, c) \mid \exists b \text{ such that } (a, b) \in R \wedge (b, c) \in S\}$