Mathematics for Computer Science

MIT 6.042J/18.062J

Deviation from the Mean.II

Chebyshev's Theorem

Random Variable R,
Given E[R] =μ, Var[R]

$$\Pr\{|R-\mu| \geq x\} \leq \frac{\text{Var}[R]}{x^2}$$

Probability that I see a value further than x away from the expected value

is limited by the variance

Another Form of Chebyshev

$$\Pr\{|R-\mu| \geq x\} \quad \leq \quad \frac{Var[R]}{x^2}$$

Let $x = c\,\sigma$, where c = constant and Var[R] = σ^2

$$\Pr\{|R-\mu| \geq c\sigma\} \quad \leq \quad \frac{Var[R]}{c^2\sigma^2}$$

$$= \frac{\cancel{\sigma^2}}{c^2\cancel{\sigma^2}} \quad = \frac{1}{c^2}$$

Interpreting Variance

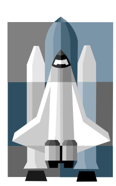
$$\Pr\{|R-\mu| \geq c\sigma\} \quad \leq \quad \frac{1}{c^2}$$

Probability that

$\lvert R-\mu\rvert \geq \sigma$	is	≤ 1	
$\lvert R-\mu\rvert \geq 2\sigma$	is	$\leq 1/4$	(25%)
$\lvert R-\mu\rvert \geq 3\sigma$	is	$\leq 1/9$	(11%)
$\lvert R-\mu\rvert \geq 4\sigma$	is	$\leq 1/16$	(6%)

Space Station Mir

Suppose that main computer has a probability p of failing every year



- Mean Time to Failure: when do we expect it to fail?

E[T] = $1/p$

Var[T] = ?

Calculating Variance

We know $\Pr\{T{=}k\} = (1-p)^{k-1}p$
How do we compute Var[T]?

$$\text{Var}[T] = \boxed{\text{E}[T^2]} - (\text{E}[T])^2$$

- Define $Y ::= T^2$
- $T = 1, 2, 3, \ldots, k, \ldots\infty$
- $Y = 1, 4, 9, \ldots, k^2, \ldots\infty$

Calculating Variance

$$E[T^2] = E[Y]$$

$$= \sum_{i \in range(Y)}^{\infty} i \Pr\{Y = i\}$$

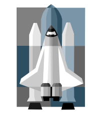
↓ Range of Y

$$= \sum_{k=1}^{\infty} k^2 \Pr\{Y = k^2\}$$

↓ Same event

$$= \sum_{k=1}^{\infty} k^2 \Pr\{T = k\} = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

Mean Time to Failure



$E[T] = 1/p$ $\quad$ $\mathrm{Var}[T] = \sigma^2 = \frac{1}{p}\left(\frac{1}{p} - 1\right)$

- $p = 1/6$ $\quad$ $E[T] = 6$, $\quad$ $\sigma = 6$ (Dice)
- $p = 1/10$ $\quad$ $E[T] = 10$, $\quad$ $\sigma = 10$ (Mir 1)
- $p = 1/1000$ $\quad$ $E[T] = 1000$, $\quad$ $\sigma = 1000$ (Mir 2)

Chebyshev tells us that probability that Mir 1 will last for more than 30 years is less than 25%

Class Exercise

$\mathrm{Var}[R] ::= E[(R - E[R])^2]$

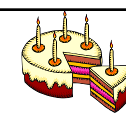
$= E[R^2] - (E[R])^2$

$\mathrm{Var}[5] = ?$

$\mathrm{Var}\,[5R] = ?$

$\mathrm{Var}[I] = ?$ where I is an indicator variable with $\Pr\{I = 1\} = p$ and $\Pr\{I = 0\} = (1 - p)$

Birthday Pairs

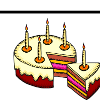


Given a room of N people, how many pairs of people have the same birthday, if a year has Y days?

$$E[P] = N^2/2Y$$

(by linearity of expectations)

Birthday Pairs



$E[P] \approx 29$ for this class (149) and planet

What is the $\Pr\{P = 23\}$?

Calculating $\Pr\{P = k\}$ is hard

However, we can still calculate the variance!

Variance of Sums

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

if X,Y are independent.

Generally,

$\mathrm{Var}[X_1 + X_2 + \cdots] =$

$\mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots$

if X_i are pairwise independent.

Pairwise Independence

U ::= Albert and Radhika have same bday

V ::= Albert and Tina have same bday

any U and V are independent

But

W ::= Radhika and Tina have same bday

($U \wedge V$), and W are NOT independent

Birthday Pairs

$X_i = 1$ if pair i has the same birthday

X_i are pairwise independent

- $E[X_i] = p$ where p ::= $1/Y$
- $Var[X_i] = p(1-p) \approx 1/Y$

Number of pairs $M \approx N^2/2$

$E[P] = E[X_1 + X_2 + \ldots + X_M] \approx$

$Var[P] = Var[X_1 + X_2 + \ldots + X_M] \approx$

Birthday Predictions

For 146 students, $E[P] \approx 29$,

$Var[P] \approx 29$ (std dev 5.3)

Chebyshev: More than 75% of the time we expect to see 29 ± 10 pairs, that is,

between 9 & 39 pairs

- What we saw: 23 pairs
- What if N is larger (larger class)?

In Class Problems

	Additivity of Variance?
• Binomial Distribution	YES
• Random Hat Check	NO

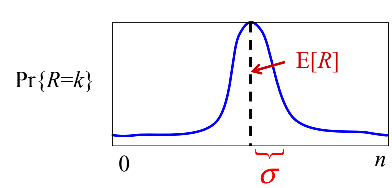
In-Class Problems

Problems 1,2,3

Binomial Distribution

$E[R] = np$

Standard Deviation $\sigma = \sqrt{np(1-p)}$



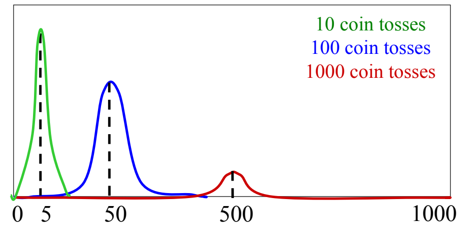
Binomial Distribution

		as a fraction of n	
$\mathrm{E}[R] = np$		p	$1/2$
$\mathrm{StdDev}[R] =$	$\sqrt{np(1-p)}$	$\sqrt{\dfrac{pq}{n}}$	$\dfrac{2}{\sqrt{n}}$

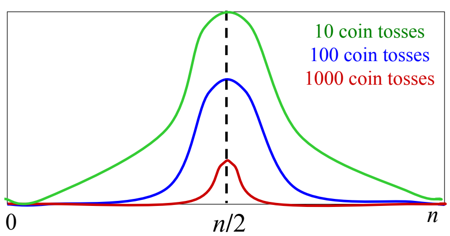
More times I toss, the smaller the variation from expected fraction of heads:

Probability of being close to the mean goes up!

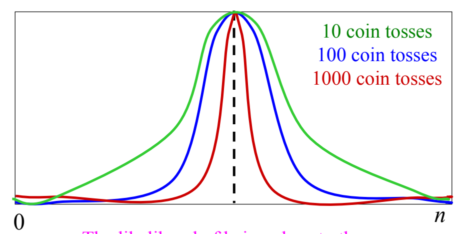
Fair Binomial Distribution



Normalize the x axis to be 0 to n



Normalize the y axis



The likelihood of being close to the mean increases the more coins we toss!

Repeated Trials

The average value of repeated experiments is likely to be close to the expected value.

And it gets more likely as we do more experiments

Weak Law of Large Numbers

Let A_n be the average of n independent experiments, each with same μ and σ

$$\lim_{n \to \infty} \Pr\{|A_n - \mu| \le \varepsilon\} = 1$$

Pairwise Independent Sampling

Let A_n be the average of n independent experiments, each with same μ and σ

$$\Pr\{|A_n - \mu| \geq x\} \leq \frac{\text{Var}[A_n]}{x^2}$$

CHEBYSHEV: The probability is limited

Applications

- Guess the bias of a coin
- Guess voting results from random polling
- Guess pollution levels from sampling Charles River water..