# In-Class Problems — Week 11, Fri

**Problem 1.** Independently flip three fair coins, and define $C$ to be the number of heads which appear and $M$ to be 1 iff all three coins match and 0 otherwise. Also, let $S ::= C \bmod 2$.

What are the values of the probability density functions (pdf's) $f_C$, $f_M$, and $f_S$? Likewise for the distribution functions $F_C$, $F_M$, and $F_S$?

**Problem 2.** Consider the binary relation $\rho$ on the set of real-valued random variables,

$$X \rho Y \longleftrightarrow \Pr\{X \neq Y\} = 0.$$

**(a)** Give an example of RV's $X, Y$ on some probability space, $\mathcal{S}$, such that $X \rho Y$ but $X \neq Y$.

**(b)** Prove that $\rho$ is an equivalence relation.

**WE DIDN'T GET TO THE FOLLOWING TWO PROBLEMS IN CLASS.**

**Problem 3.** In addition to $C, M$, and $S$, in problem 1, let $R_i$ be the indicator variable for a Head occurring on the $i$th flip, for $i = 1, 2, 3$. Verify that $R_1, R_2, R_3, S$ are 3-way independent, that $R_1, R_2, R_3, S, M$, are pairwise independent, and that there are no other pairs of independent variables besides these.

**Problem 4.** Let $X, Y$ be independent Binomial random variables with parameters $(n, p)$ and $(m, p)$, respectively.

**(a)** What is $\Pr\{X + Y = k\}$?

**(b)** Prove that if $p = 1/2$ and $n$ is even, then

$$\Pr\left\{X = \frac{n}{2}\right\} \sim \sqrt{\frac{2}{n\pi}}. \tag{1}$$

*Hint:* Use Stirling's approximation (in the appendix).

**(c)** Estimate the probability that the number of heads in 400 flips of a fair coin will be between 195 and 205, and likewise that in 10,000 flips it will be between 4980 and 5020.

Also, discuss writing a program to calculate the exact answer.

## Random Variables

A *random variable* over a given sample space, $\mathcal{S}$, is a function from $\mathcal{S} \to \mathbb{R}$.

The *probability density function (pdf)* for a random variable, $R$, is the function $f_R : \text{range}(R) \to [0, 1]$ defined by:

$$f_R(x) ::= \Pr\{R = x\}.$$

For any event $A$, its *indicator variable*, $I_A$, is the 0-1 valued variable such that the event $[I_A = 1]$ is the same as the event $A$. It follows that $[I_A = 0]$ is the same as $\overline{A}$.

A random variable, $U$, is *uniform* iff all its values are equally likely. That is

$$\Pr\{U = a\} = \Pr\{U = b\}$$

for all $a, b \in \text{range}(U)$. In other words, its pdf is constant.

A random variable, $B$, is *binomial* with parameters $(n, p)$ iff its pdf is the function $f_{n,p} : \mathbb{N} \to [0, 1]$ defined by

$$f_{n,p}(k) ::= \binom{n}{k} p^k (1-p)^{n-k}$$

where parameter $n \in \mathbb{N}$ and $0 < p < 1$. Equivalently, $B = \sum_{k=1}^{n} B_k$ where $B_1, B_2, \ldots, B_n$ are mutually independent Bernoulli (indicator) variables with $\Pr\{B_i = 1\} = p$.

Random variables $R_1, R_2, \ldots$ are *mutually independent* iff

$$\Pr\left\{\bigcap_i [R_i = x_i]\right\} = \prod_i \Pr\{R_i = x_i\},$$

for all $x_1, x_2, \dots \in \mathbb{R}$. They are *k-wise independent* iff $\{R_i \mid i \in J\}$ are mutually independent for all subsets $J \subset \mathbb{N}$ with $|J| = k$.

## Stirling's Approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$