



Combinatorics II.3



Multisets

Definition. A *multiset* is a collection of objects, some of which may not be distinguishable from each other.

Example. $\{a, b, a, c, b, a\} = \{3 \cdot a, 2 \cdot b, 1 \cdot c\}$

repetition counts



Permutations of Multisets

Let S be a multiset with k objects, each with an infinite repetition count. That is,

$$S = \{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_k\}.$$

Q. How many r -permutations of S are there?

A. k^r (easy).



Permutations of Multisets with Limited Repetitions

Q. How many permutations are there of **MISSISSIPPI** ?

$$(S = \{ 1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P \} .)$$

A. **IDEA:** Fill in slots.



Permutations of Multisets with Limited Repetitions

$$S = \{ 1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P \}$$

$$\begin{aligned} \# \text{ choices for M: } & \binom{11}{1} \\ \times \# \text{ choices for I: } & \binom{10}{4} \\ \times \# \text{ choices for S: } & \binom{6}{4} \\ \times \# \text{ choices for P: } & \binom{2}{2} \end{aligned} \quad 34,650$$



Multinomial Coefficients

Definition. For $n = r_1 + r_2 + \dots + r_k$, the **multinomial coefficient** $C(n; r_1, r_2, \dots, r_k)$ is defined as

$$\begin{aligned} \binom{n}{r_1, r_2, \dots, r_k} &= \binom{n}{r_1} \cdot \binom{n-r_1}{r_2} \cdot \binom{n-r_1-r_2}{r_3} \dots \binom{n-r_1-r_2-\dots-r_{k-1}}{r_k} \\ &= \frac{n!}{r_1! r_2! \dots r_k!} \end{aligned}$$



Permutations of a Multiset with Limited Repetitions

Theorem. The number of permutations of a multiset $\{r_1 \cdot a_1, r_2 \cdot a_2, \dots, r_k \cdot a_k\}$, where $n = r_1 + r_2 + \dots + r_k$ is

$$\binom{n}{r_1, r_2, \dots, r_k}.$$

Proof. Fill in the n slots, as before. \square



In-Class Problems

Problems 1 & 2



Combinations of Multisets with Unlimited Repetitions

Q. How many 2-combinations can be chosen from the multiset

$$S = \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}?$$

A. 10:

$$\{a, a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, b\}, \{b, c\}, \{b, d\}, \{c, c\}, \{c, d\}, \{d, d\}.$$



Stars-and-Bars Theorem

Theorem. The number of r -combinations of a multiset with n distinct objects, each with ∞ repetition count, is

$$\binom{n-1+r}{r}.$$

Proof (Stars and Bars). Define a bijection from the r -combinations of $S = \{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$ to the permutations of $T = \{r \cdot \star, (n-1) \cdot | \}$.



Establishing a Bijection

An r -combination of $S = \{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$ looks like

where $r = r_1 + r_2 + \dots + r_n$.

The corresponding permutation of $T = \{r \cdot \star, (n-1) \cdot | \}$ looks like

$\underbrace{\star\star\star}_r | \underbrace{\star\star}_r | \underbrace{|}_r \dots | \underbrace{\star\star\star}_r .$

Double-check: one-to-one, onto.



Example Bijection

How many 2-combinations from $S = \{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$?

- We have 2 \star 's and $4-1=3$ $|$'s:

$\star\star||| = \{a, a\}$ $| \star | \star | = \{b, c\}$
 $\star | \star || = \{a, b\}$ $| \star || \star = \{b, d\}$
 $\star || \star | = \{a, c\}$ $| | \star \star | = \{c, c\}$
 $\star || | \star = \{a, d\}$ $| | \star | \star = \{c, d\}$
 $| \star \star || = \{b, b\}$ $| | | \star \star = \{d, d\}$



Finishing the Proof

Each permutation of $T = \{r \cdot \star, (n-1) \cdot | \}$ can be enumerated by considering $n-1+r$ slots, each filled by a \star or a $|$, and then choosing r of the slots to hold $|$'s:

$\star\star\star | \star\star | | \dots | \star\star\star .$

Thus, we have

r -combinations of $S =$ # permutations of T

$$= \binom{n-1+r}{r}. \quad \square$$



In-Class Problems

Problem 3