Mathematics for Computer Science

MIT 6.042J/18.062J

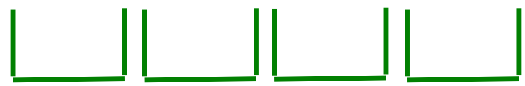
Combinatorics I.2

Pigeons and Holes

Pigeonhole Principle

If you have *more* pigeons

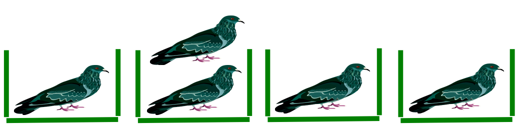
than pigeonholes,



Pigeonhole Principle

Then some hole

must have *two* pigeons!



Example: 5 Card Draw

Pick a set of 5 cards.

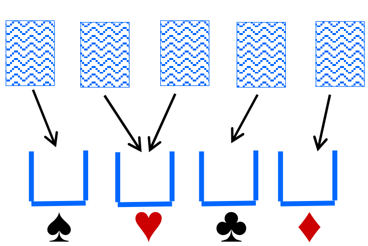
Guaranteed to have

2 with the *same suit*.

5 Card Draw

5 cards (pigeons)

4 suits (holes)

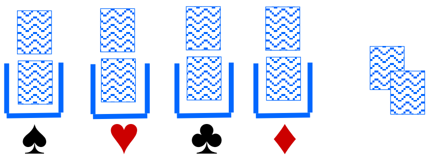


10 Card Draw

How many cards of the same suit am I guaranteed to have?

$$\left\lceil \frac{10}{4} \right\rceil = 3$$

10 Card Draw



Cannot have < 3 cards in every hole

Generalized Pigeonhole Principle

If there are n pigeons and t holes, then there will be at least one hole with at least

$$\left\lceil \frac{n}{t} \right\rceil \text{ pigeons}$$

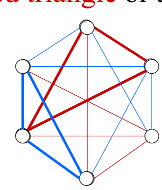
A Pigeonhole Problem

Six people.
Every two are either friends or strangers.

There must be a set of
3 mutual friends or 3 mutual strangers

Restated as a Graph problem

In a 6-node complete graph where every edge is colored red or blue, there always exists *either* a red triangle or a blue triangle

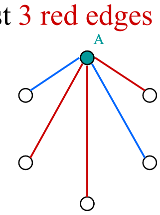


VERIFY THIS!

…by constructing a graph for your table.

Proof

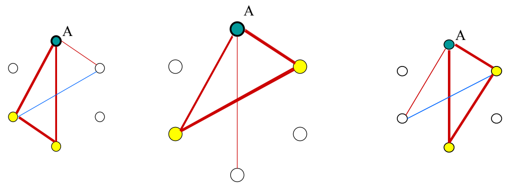
Vertex A has degree 5
Pigeonhole Principle:
A has at least 3 red edges or 3 blue edges



Proof

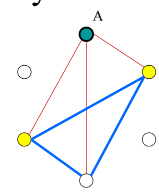
Let A have ≥ 3 red edges.

If ends of any two edges are connected by a red edge, then a red triangle is formed



Proof

Otherwise, they are connected by blue edges



Ramsey Numbers

Theorem: For any natural number k, there exists a n such that a red-blue complete graph on n nodes must have a size k complete blue or red subgraph

$R(k,k)$ ::= smallest such n

$R(3,3) = 6$

R(6,6) = ?

How many people do you need before you are guaranteed to have a set of six friends or six strangers?

Paul Erdös's answer: If aliens required you answer this in a year, throw in the towel and strike preemptively….

In-Class Problems

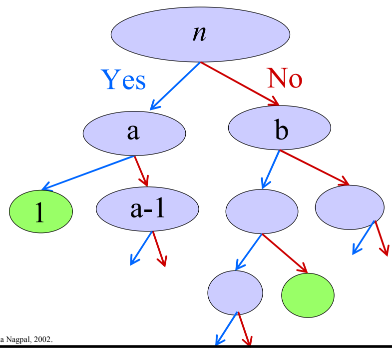
Problems 1 & 2

Pick a Number Game

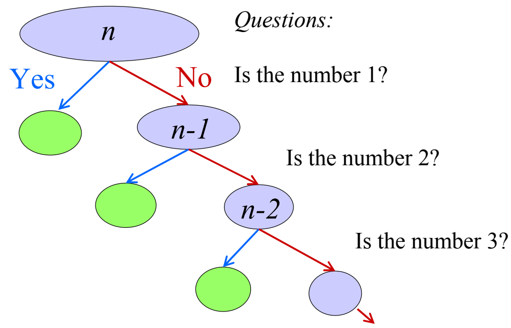
- *Player 1:* Pick a number 1 to Million
- *Player 2:* Can ask Yes/No questions

- How many questions do I need to ask in order to be guaranteed to correctly identify the number?

A Strategy as a Tree



A Possible Strategy



Minimum Number of Questions

- How many questions do I need to ask to guess the number?
  - Worst case, it's the height of the strategy tree
- What is the minimum number of questions I could ask and be guaranteed to find out the number?

Pigeonhole Principle

If a tree has < n leaves, then there are at least 2 numbers that can not be distinguished.

- The strategy tree must have at least *n* leaves

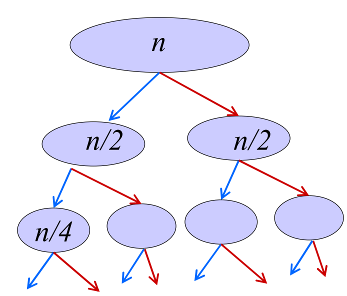
Shortest Strategy Tree

What is the shortest binary tree with *n* leaves?

- Full Binary Tree
- Height = $\log_2 n$

Lower bound on the number of questions = $\log_2 n$

Optimal Strategy



Lower Bounds

- How many weighings do you need to find one counterfeit coin amongst 12 coins?
- How many binary comparisons do you need to sort n numbers

Lower Bound on Sorting

- What is the minimum number of binary comparisons needed to *sort* n numbers?
- = guessing which permutation of the n numbers you got
- = need $n!$ leaves

In-Class Problems

Problem 3

Dilworth's Theorem

If poset with n elements

has a longest chain of size t,

then there exists an antichain of

size at least $\left\lceil \frac{n}{t} \right\rceil$

Proof

If longest chain has size t,

then the n elements can be partitioned

into t antichains *(parallel task scheduling thm)*

So have t antichains (*holes*)

and n elements (*pigeons*):

there must exist at least one antichain with $\left\lceil \frac{n}{t} \right\rceil$ elements *(pigeonhole principle)*