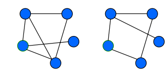
Mathematics for Computer Science

MIT 6.042J/18.062J

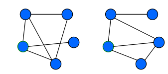
Combinatorics I

Counting in Graph Theory

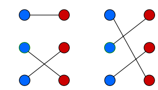
- How many different *n*-node graphs are there?



- How many different mappings need to be checked to see if two arbitrary *n*-node graphs are isomorphic?



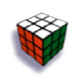
- How many different pairings between *n* boys and *n* girls are there?



Counting in Algorithms

- What is the minimum number of binary comparisons needed to *sort* *n* numbers?
- What is the fastest any algorithm could possibly sort?

Counting in Games



- How many different configurations exist for a Rubik's cube?

- How many different chess positions can exist after *n* moves?

- How many weighings are needed to find the one counterfeit coin among 12 coins?

Counting in Probability

Probability of an *event* in a uniform sample space:

$$\frac{\text{\# event outcomes}}{\text{total \# outcomes}}$$

Counting in Probability

What is the probability of getting two jacks in a poker hand?

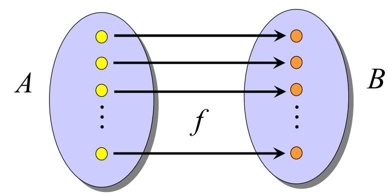
Counting Techniques

- Bijections
- Sum Rule, Product Rule
- Inclusion-Exclusion
- Pigeonhole Principle
- Trees
- Permutations

Bijections

If f is a ***bijection*** from A to B, then

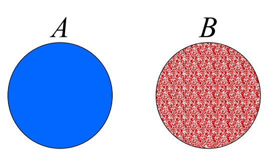
$$|A| = |B|$$



Sum Rule

If sets A and B are disjoint, then

$$|A \cup B| = |A| + |B|$$



Product Rule

If $|A| = m$ and $|B| = n$, then $|A \times B| = mn$.

$A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$

$A \times B = \{$ (a,1), (a,2), (a,3),
(b,1), (b,2), (b,3),
(c,1), (c,2), (c,3),
(d,1), (d,2), (d,3) $\}$

Product Rule: Counting Strings

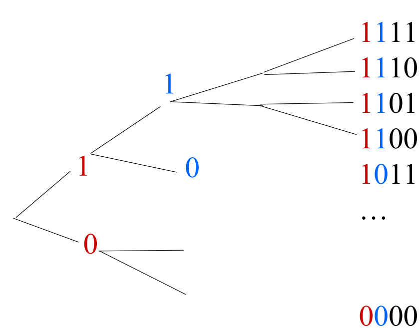
The number of length-4 strings from the alphabet $A = \{0, 1\}$

$$= |A \times A \times A \times A|$$

$$= 2 \cdot 2 \cdot 2 \cdot 2$$

$$= 2^4$$

Product Rule: Tree



Product Rule: Counting Strings

The number of length-n strings from an alphabet A of size m is

$$m^n.$$

Example

Given a finite set A, how many subsets of A are there? That is, what is $|P(A)|$, where

$P(A)$ = the ***power set*** of A

= the set of all subsets of A?

$A = \{a, b, c\}$

$P(A) = \{ \varnothing,$
$\{a\}, \{b\}, \{c\},$
$\{a,b\}, \{b,c\}, \{c,a\},$
$\{a,b,c\} \}$

Bijection: P(*A*) and Binary Strings

$A = \{a_1, a_2, a_3, a_4, a_5, \ldots, a_n\}$

String = 1 0 1 1 0 … 1

Subset = $\{a_1, \quad a_3, a_4, \quad \ldots, \quad a_n\}$

Exact correspondence:

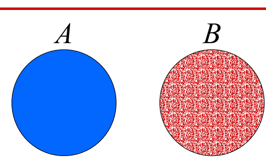
$2^n = |n\text{-bit binary strings}| = |P(A)|$

In-Class Problem

Problem 1

Sum Rule

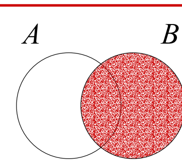
If sets A and B are disjoint, then
$|A \cup B| = |A| + |B|$



What if A and B are not disjoint?

Inclusion-Exclusion (2 Sets)

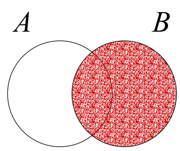
For two arbitrary sets A and B
$$|A \cup B| = |A| + |B| - |A \cap B|$$



Inclusion-Exclusion (2 Sets)

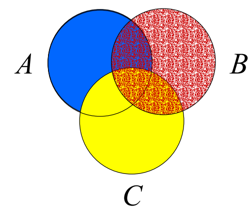
Corollary. ***Boole's Law:***

$$|A \cup B| \leq |A| + |B|$$



Inclusion-Exclusion (3 Sets)

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Inclusion-Exclusion (n Sets)

$|A_1 \cup A_2 \cup \cdots \cup A_n| =$

sum of sizes of all single sets

– sum of sizes of all 2-set $\cap$'s

+ sum of sizes of all 3-set $\cap$'s

– sum of sizes of all 4-set $\cap$'s

⋮

+ $(-1)^{n+1}$ · sum of sizes of $\cap$ of all n sets

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{S \subseteq \{1,2,\ldots,n\} \\ |S|=k}} \left| \bigcap_{i \in S} A_i \right|$$

Inclusion-Exclusion (n Sets)

Alternative formulation:

$$|A_1 \cup A_2 \cup \cdots \cup A_n|$$

$$= \sum_{\emptyset \neq S \subseteq \{1,2,..,n\}}^{n} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$

In-Class Problem

Problem 2