

Solutions to In-Class Problems — Week 3, Mon

Problem 1. Find the flaw in the following false proof, and give a counter-example to the claim.

Claim. *Suppose R is a relation on A . If R is symmetric and transitive, then R is reflexive.*

False proof. Let x be an arbitrary element of A . Let y be any element of A such that xRy . Since R is symmetric, it follows that yRx . Then since xRy and yRx , we conclude by transitivity that xRx . Since x was arbitrary, we have shown that $\forall x \in A (xRx)$, so R is reflexive. \square

Solution. The flaw is assuming that y exists. It is possible that there is an $x \in A$ that is not related by R to anything. No such R will be reflexive. The simplest such R that is also symmetric and transitive is the empty relation on any nonempty set A . We can easily construct other examples, such as

$$R = \{(a, a), (a, b), (b, a), (b, b)\}$$

on the set $A ::= \{a, b, c\}$, which is not reflexive because (c, c) is not in R . These relations are counterexamples to the claim.

Note that the theorem can be fixed: R restricted to its domain of definition is reflexive, and hence an equivalence relation. \blacksquare

Problem 2. In each case, say whether or not R is an equivalence relation on A . If it is an equivalence relation, what are the equivalence classes and how many equivalence classes are there?

(a) $R ::= \{(x, y) \in W \times W \mid \text{the words } x \text{ and } y \text{ start with the same letter}\}$ where W is the set of all words in the 2001 edition of the Oxford English dictionary.

Solution. R is an equivalence relation since it is reflexive, symmetric, and transitive. The equivalence class of x with respect to R is the set $[x]_R = \{y \mid y \text{ has the same first letter as } x\}$. There are 26 equivalence classes, one for each letter of the English alphabet. \blacksquare

(b) $R ::= \{(x, y) \in W \times W \mid \text{the words } x \text{ and } y \text{ have at least one letter in common}\}$.

Solution. S is reflexive and symmetric, but it is not transitive. Therefore, S is not an equivalence relation. For example, let w_1 be the word “scream,” let w_2 be the word “and,” and let w_3 be the word “shout.” Then w_3Sw_1 , and w_1Sw_2 , but it is **not** the case that w_3Sw_2 . ■

(c) $R = \{(x, y) \in W \times W \text{ and the word } x \text{ comes before the word } y \text{ alphabetically}\}$.

Solution. R is not reflexive but it is transitive and antisymmetric. It is not an equivalence relation, but it is a partial order. ■

(d) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \text{ and } |x| \leq |y|\}$.

Solution. R is reflexive and transitive. It is not symmetric. It is not antisymmetric either. As a counterexample, $|-3| \leq |3|$, $|3| \leq |-3|$, but $3 \neq -3$. ■

(e) $R = \{(x, y) \in B \times B, \text{ where } B \text{ is the set of all bit strings and } x \text{ and } y \text{ have the same number of 1s.}\}$

Solution. R is reflexive, symmetric and transitive, and therefore an equivalence relation. There is an equivalence class for each natural number corresponding to bit strings with that number of 1s. ■

Problem 3. Let R be an equivalence relation on the set A . For an element $a \in A$, let $[a]$ denote the set $\{b \in A \text{ given } aRb\}$. This set is the *equivalence class of a under R* and we call a a *representative of the set $[a]$* .

Prove that the sets $[a]$ for all $a \in A$ constitute a partition of A . In other words, prove that for every $a, b \in A$, either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

Solution. *Proof.* Consider some arbitrary $a, b \in A$. If either $[a] = [b]$ or $[a] \cap [b] = \emptyset$ then we are done, so suppose not.

Let c be any element that is in one of the sets but not the other. Without loss of generality we can assume that $c \in [b] - [a]$. (We know that either $c \in [b] - [a]$ or $c \in [a] - [b]$. In the latter case we can simply swap a and b and reduce to the first case.)

Let d be any element in $d \in [a] \cap [b]$. We will get a contradiction by showing that aRc and therefore that $c \in [a]$. First, aRd because $d \in [a]$ (note that $d = a$ is a possibility but this is ok because R is reflexive). Second, dRb and bRc because both $c, d \in [b]$ and R is symmetric. This implies, by transitivity, that dRc . Finally, by transitivity, aRc because aRd and dRc . Hence we have a contradiction. Also, we know that for every $a \in A$, a is in its own equivalence class since R is

reflexive. Therefore the union of all the equivalence classes gives us A , and thus the sets $[a]$ for $a \in A$ constitute a partition of A .

Note that all three properties of equivalence relations were used in this proof. Checking that the proof uses all available assumptions is usually a good sanity check when writing proofs. If one of the properties you assumed were not needed, you have either made a mistake or proven a much more strong theorem than you thought. For example, if you didn't use transitivity anywhere in the proof of this problem, you would have falsely proven that any reflexive symmetric relation produces a partition. \square

Appendix

A binary relation R on a set A is

- *reflexive* if for every $a \in A$, $a \sim_R a$,
- *symmetric* if for every $a, b \in A$, $a \sim_R b$ implies $b \sim_R a$.
- *antisymmetric* if for every $a, b \in A$, $a \sim_R b$ and $b \sim_R a$ implies $a = b$.
- *asymmetric* if for every $a, b \in A$, $a \sim_R b$ implies $\neg(b \sim_R a)$.
- *transitive* if for every $a, b, c \in A$, $a \sim_R b$ and $b \sim_R c$ implies $a \sim_R c$.

for all $x, y, z \in A$.

A binary relation R on a set A is

- an *equivalence relation* iff it is reflexive, symmetric and transitive.