Laws of Probability

Probability Spaces

1) Sample space, $\mathcal{S}$, whose elements are called outcomes.
2) Probability function,
   Pr: $\mathcal{S} \to [0,1]$
   (a) $\Pr\{\mathcal{S}\} = 1$,
   (b) the Sum Rule:

Basic Laws of Probability

Sum Rule

$$\Pr\{A_1 \cup A_2\} = \Pr\{A_1\} + \Pr\{A_2\}$$

for $A_1 \cap A_2 = \emptyset$.

Sum Rule for Sets

Corresponding Rule for Sets:

$$|A_1 \cup A_2| = |A_1| + |A_2|$$

for $A_1 \cap A_2 = \emptyset$.

Difference Rule

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\}$$

Proof:

A is the disjoint union of $A - B$ and $A \cap B$.

Complement Rule

$$\Pr\{\bar{B}\} = 1 - \Pr\{B\}$$

Proof:

$\Pr\{\bar{B}\} = \Pr\{\mathcal{S} - B\}$

$= \Pr\{\mathcal{S}\} - \Pr\{\mathcal{S} \cap B\}$ (Diff Rule)

$= 1 - \Pr\{B\}$.

Inclusion-Exclusion

$$\Pr\{A \cup B\} =$$
$$\Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\}$$

Boole's Inequality

$$\Pr\{A \cup B\} \leq \Pr\{A\} + \Pr\{B\}$$

Monotonicity

$$\Pr\{A\} \leq \Pr\{A \cup B\}$$

Basic Laws of Probability

Infinite Sum Rule

$$\Pr\{A_1 \cup A_2 \cup \cdots\}$$
$$= \Pr\{A_1\} + \Pr\{A_2\} + \cdots$$

for pairwise disjoint A_n.

$$\Pr\{\bigcup_{n \in N} A_n\} = \sum_{n \in N} \Pr\{A_n\}$$

Boole's Inequality

$$\Pr\{\bigcup_{n \in N} A_n\}$$
$$\leq \sum_{n \in N} \Pr\{A_n\}$$

Boole's Inequality

Example: 10,000 transister chip with Pr{transister failure} ≈ 1/1,000,000.
Chip fails if any transister fails.

Pr{Chip fails}

= Pr{ ∪ [*i*th transister fails]}

≤ Σ Pr{*i*th transister fails}

≈ (1/M)·(10,000) = 0.01

Boole's Inequality

So Pr{Chip works} ≈ 0.99, without any assumptions on relations between transister failures, e.g., whether all chips fail together, or never fail together, or are unrelated.

In-class Problem

Problem 1

Probability Analysis Method

1. Identify outcomes (*tree helps*)
2. Identify event (*e.g. winning*)
3. Assign outcome probabilities
4. Compute event probabilities (*sum outcome probabilities*)

Assigning Outcome Probabilities

Often reason from symmetry:

- one 5-card hand as likely as another,
- Heads as likely as Tails,
- Pr{hit radius r target} = (1/4) · Pr{hit radius $2r$ target}
- Monty Hall tree structure

Assigning Outcome Probabilities

Also reason from data:

- Pr{family pet is a cat} ≈ 0.69
- Pr{Massachusetts voter is Republican} ≈ 0.43

Assigning Outcome Probabilities

But probability assignment is an axiom. Nothing is *mathematically* wrong with bad choices, eg, uniform Monty Hall outcomes, or very skewed probabilities for Heads versus Tails. Conclusions based on bad assignments will be *perfectly consistent*, they just won't predict reality well.

Penny Flipping Experiment

- Balance 40 pennies on their edges. (Separately!)
- Pound the table till all fall.
- Record h ::= #heads, t ::= #pennies.
- Repeat at least 3 times.

The End

Lecture ended here without getting to subsequent slides.

Conditional Probability

"Knowledge" changes probabilities:

Pr{die rolled 1} = 1/|{1,2,3,4,5,6}|
= 1/6.

Pr{die rolled 1 knowing that die rolled odd number}
= 1/|{1,3,5}|
= 1/3.

Conditional Probability

$\Pr\{A \mid B\}$ ::=
probability of event A *given* that event B has occurred.
Formally,

$$\Pr\{A \mid B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

Conditional probability for Monty Hall

Pr{prize at Door 1 | Carol opens 2}
= 1/2. Really!

$$[\text{Carol opens 2}] = \{\underbrace{(1,1,2)}_{\Pr=\frac{1}{18}}, \underbrace{(1,3,2)}_{\Pr=\frac{1}{9}}, (3,3,2), (3,1,2)\}$$

Conditional probability for Monty Hall

This suggests the contestant may as well stick, since the probability is 1/2 *given what he knows* when he gets to stick or switch?

Not so: Contestant knows *more* than door opened by Carol -- also knows which door he chose himself!

Conditional probability for Monty Hall

Pr{prize at Door 1 | Contestant chose 1 & Carol opens 2}
= 1/3.

[Contestant chose 1 & Carol opens 2]

$$= \{\underbrace{(1,1,2)}_{\Pr=\frac{1}{18}}, \underbrace{(3,1,2)}_{\Pr=\frac{1}{9}}\}$$

Product Rule

$$\Pr\{A \cap B\}$$
$$= \Pr\{A \mid B\} \Pr\{B\}$$

In-class Problem

Problems 2&3