

## Solutions to In-Class Problems — Week 11, Fri

**Problem 1.** Independently flip three fair coins, and define  $C$  to be the number of heads which appear and  $M$  to be 1 iff all three coins match and 0 otherwise. Also, let  $S ::= C \bmod 2$ .

What are the values of the probability density functions (pdf's)  $f_C$ ,  $f_M$ , and  $f_S$ ? Likewise for the distribution functions  $F_C$ ,  $F_M$ , and  $F_S$ ?

**Solution.**

$$f_C(0) = f_C(3) = 1/8$$

$$f_C(1) = f_C(2) = 3/8$$

$$f_M(0) = 3/4$$

$$f_M(1) = 1/4$$

$$f_S(0) = f_S(1) = 1/2.$$

$$F_C(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1/8 & \text{for } 0 \leq x < 1, \\ 1/2 & \text{for } 1 \leq x < 2, \\ 7/8 & \text{for } 2 \leq x < 3, \\ 1 & \text{for } 3 \leq x. \end{cases}$$

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**Problem 2.** Consider the binary relation  $\rho$  on the set of real-valued random variables,

$$X \rho Y \iff \Pr \{X \neq Y\} = 0.$$

(a) Give an example of RV's  $X, Y$  on some probability space,  $\mathcal{S}$ , such that  $X \rho Y$  but  $X \neq Y$ .

**Solution.** Let

$$\begin{aligned} \mathcal{S} &::= \{\text{win}, \text{lose}\}, \\ \Pr\{\text{win}\} &::= 1, \\ \Pr\{\text{lose}\} &::= 0, \\ X(\text{win}) = Y(\text{win}) &::= 5, \\ X(\text{lose}) &::= 5, \\ Y(\text{lose}) &::= 4.99. \end{aligned}$$

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(b) Prove that  $\rho$  is an equivalence relation.

**Solution.** The trick is to observe that

$$\begin{aligned} \Pr\{X \neq Y\} = 0 &\iff \Pr\{X = Y\} = 1 \\ &\iff X(w) = Y(w) \\ &\text{for all outcomes } w \in \mathcal{S} \text{ such that } \Pr\{w\} \neq 0. \end{aligned}$$

Now Reflexivity and Symmetry are immediate. To prove Transitivity, suppose  $\Pr\{X = Y\} = 1 \wedge \Pr\{Y = Z\} = 1$ . So,  $X(w) = Y(w)$  and  $Y(w) = Z(w)$  for all  $w$  such that  $\Pr\{w\} \neq 0$ . By transitivity of equality,  $X(w) = Z(w)$  for all  $w$  such that  $\Pr\{w\} \neq 0$ . Hence,  $\Pr\{X = Z\} = 1$ , as required.

An alternative proof of transitivity that works for general probability spaces—not just the “discrete” spaces considered in 6.042—makes ingenious use of Inclusion-Exclusion:

$$\Pr\{X = Z\} \geq \Pr\{X = Y \wedge Y = Z\} \quad (\text{monotonicity}) \quad (1)$$

$$= \Pr\{X = Y\} + \Pr\{Y = Z\} - \Pr\{X = Y \vee Y = Z\} \quad (\text{inclusion-exclusion}) \quad (2)$$

$$\geq 1 + 1 - 1 = 1. \quad (3)$$

So  $\Pr\{X = Z\} = 1$ . ■