Mathematics for Computer Science

MIT 6.042J/18.062J

Partial Orders

Types of Relations

Equivalence

- Reflexive, Transitive, *Symmetric*.

Partial Orders

- (Reflexive), Transitive, *Antisymmetric*.

Antisymmetric

$$\forall a,b \quad a \neq b \wedge aRb \rightarrow \neg(bRa)$$

Maybe *aRb* or *bRa*, **never both.**

– Either places an order on *a,b*

– Or *a* and *b* are unrelated (*incomparable*)

Partial Order (Poset (Z,≤))

$a\ R\ b$ if $a \leq b,\ a,b \in Z$

- Reflexive YES
- Transitive YES
- Symmetric NO
- Antisymmetric YES

- in fact a Total Order

Total Order

- A relation *R* on a set *A* is a total order if it is a partial order and for any *a,b* in *A*, either *aRb* or *bRa*

– For any two distinct integers, *x* and *y*, either *x*<*y* or *y*<*x*

Not a Total Order: Divisibility

$a\ R\ b$ if $a \mid b,\ \ a,b \in Z$

- 3|9, but not 9|3
- But 5 and 9 are not related (incomparable)

Symmetric vs Antisymmetric

Symmetric:

$$\forall a,b \quad aRb \rightarrow bRa$$

Antisymmetric:

$$\forall a,b \quad a \neq b \wedge aRb \rightarrow \neg(bRa)$$

EXERCISE

- *Question:* Can R be neither symmetric nor antisymmetric?
- YES: $R ::= \{(1,2)\ (2,1)\ (1,4)\}$

- *Question:* Can R be both symmetric and antisymmetric?
- YES: $R ::= \{(1,1)\}$

A Relation on Classes

cRd ::= class *c* is *listed* as a prerequisite to class *d* in the 6-3 curriculum

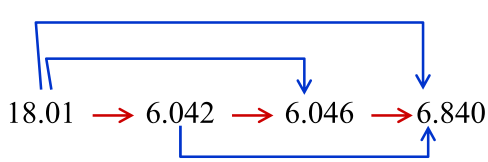
A Partial Order?

cRd ::= class *c* is *listed* as a prerequisite to class *d* in the 6-3 curriculum

- Reflexive No
- Transitive No
- Antisymmetric Yes *(certainly hope so!)*

Creating a Partial Order

- Reflexive and Transitive closure of R



A Relation on Classes

cRd ::= class *c* is *listed* as a prerequisite to class *d* in the 6-3 curriculum

Prerequisites

- 18.01 → 6.042
- 18.01 → 18.02
- 18.01 → 18.03
- 8.01 → 8.02
- 6.001 → 6.034
- 6.042 → 6.046
- 18.03, 8.02 → 6.002
- 6.001, 6.002 → 6.004
- 6.001, 6.002 → 6.003
- 6.004 → 6.033
- 6.033 → 6.857
- 6.046 → 6.840

Scheduling Problems

- How many terms will it take to graduate?
- How many classes does one need to take each term?
- Who's going to plan the whole thing?

- build a *Dependency Graph*

Identify Minimal Elements

- **18.01** → 6.042
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Directed Acyclic Graph (DAG)

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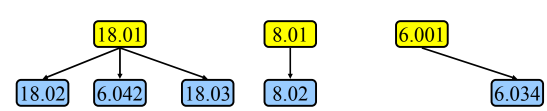

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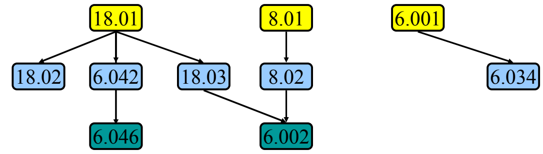

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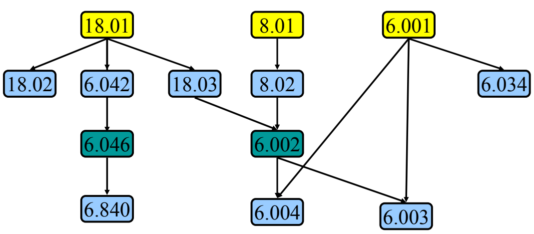

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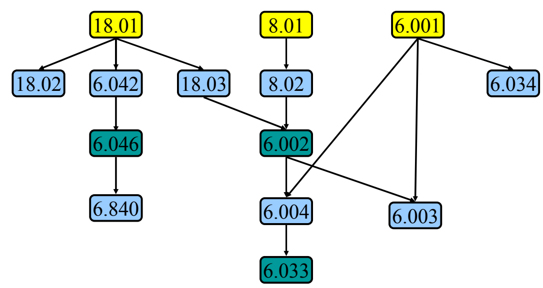

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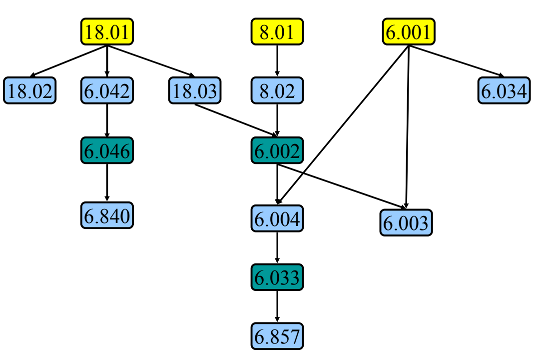
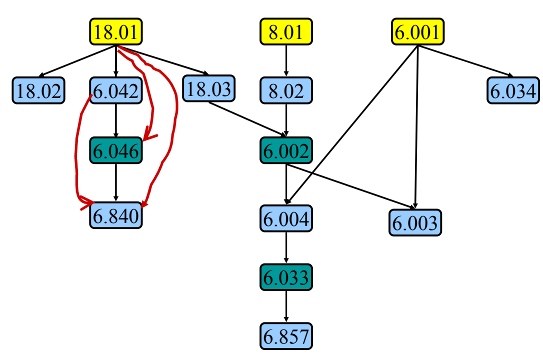
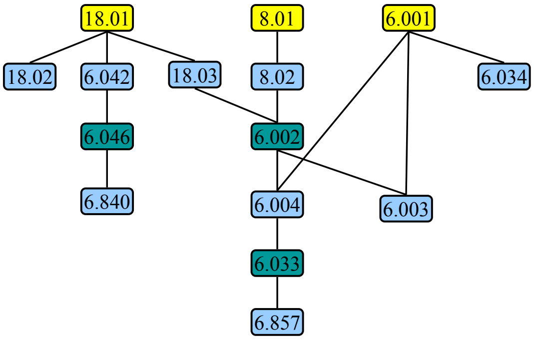


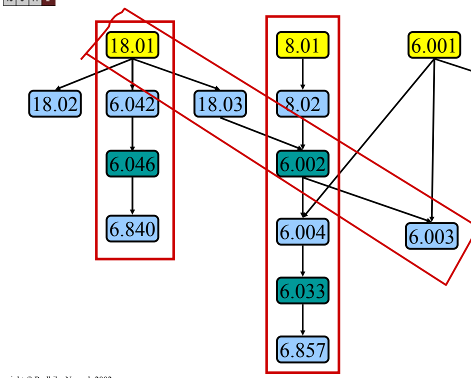
Partial Order (transitive edges)



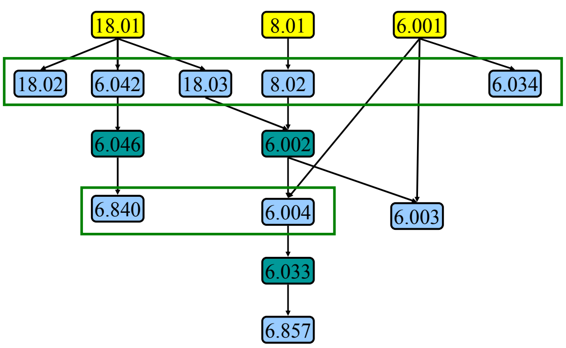
Hasse Diagram (no transitive edges)



Chains



Anti Chains



In-Class Problem 1

Scheduling Problems

- How many terms will it take to graduate?
- How many classes does one need to take each term?
- Who's going to plan the whole thing?

Parallel Task Scheduling

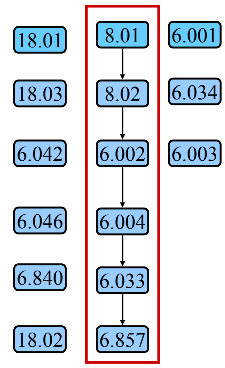
Theorem: If the longest chain has size t, then the elements can be partitioned into t antichains.

- 6 terms are **necessary** to complete the curriculum
- *and* **sufficient** (if you can take unlimited courses per year...)

Prereqs Graph

- Total **15** classes
- Length of maximum chain = **6**
- No chain of length 7
- → must be ***at least one antichain*** with size at least **3**.
  - *(at least one term where you have to take ≥3 classes)*

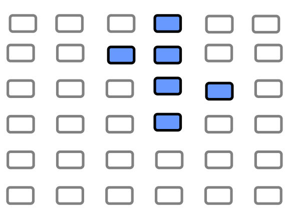
Classes per Term



Dilworth's Theorem: Example

Consider a set S, where $|S| = n = 6$

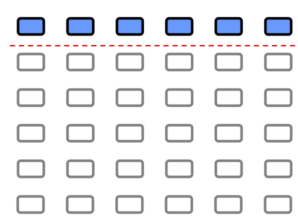
Column= chain, Row = anti-chain



Dilworth's Theorem: Example

$|S| = n = 6$

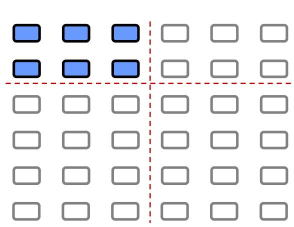
Longest chain $t = 1$



Dilworth's Theorem: Example

$|S| = n = 6$

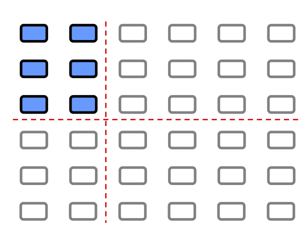
Longest chain $t = 2$



Dilworth's Theorem: Example

$|S| = n = 6$

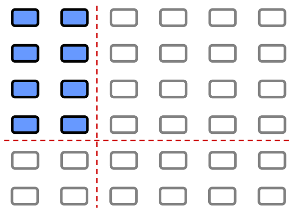
Longest chain $t = 3$



Dilworth's Theorem: Example

$|S| = n = 6$

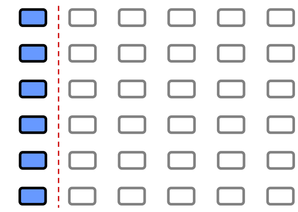
Longest chain $t = 4$



Dilworth's Theorem: Example

$|S| = n = 6$

Longest chain $t = 6$



Dilworth's Theorem

Theorem: Every poset of n elements has

- Either a chain of size at least t,
- or an antichain of size at least $\left\lceil \frac{n}{t} \right\rceil$

for all $1 \le t \le n$.

In-Class Problems