In-Class Problems — Week 13, Mon

Problem 1. Consider the following two gambling games.

Game A: We win \$2 with probability $2/3$ and lose \$1 with probability $1/3$.

Game B: We win \$1002 with probability $2/3$ and lose \$2001 with probability $1/3$.

(a) What is the expected win in each case?

(b) What is the variance in each case?

Problem 2. Suppose you have learned that the average graduating MIT student's total number of credits is 200.

(a) Knowing only this average, use Markov's inequality to find a best possible upper bound for the fraction of MIT students graduating with at least 235 credits. [1]

(b) Demonstrate that this is a best possible bound by giving a distribution for which this bound holds with equality.

(c) Suppose you are now told that no student can graduate with fewer than 170 units. How does this allow you to improve your previous bound? As before, show that this is the best possible bound.

(d) Now suppose you *further* learn that the standard deviation of the total credits per graduating student is 7. Give a best possible bound on the fraction of students who can graduate with at least 235 credits.

Problem 3. In this problem we will derive Chebyshev's Theorem from a corollary of Markov's Theorem.

(a) Explain why the following corollary of Markov's theorem holds:

[1] Ignore the fact that there are practical limits to the amount of time a student can stay at MIT and remain sane; That is, assume that there is no bound on the number of credits a student may earn.

Corollary. *For any random variable R, any positive integer k, and any $x > 0$,*

$$\Pr\{|R| \geq x\} \leq \frac{\mathrm{E}\left[|R|^k\right]}{x^k}.$$

(b) Use the above corollary to prove the following:

Theorem (Chebyshev). *Let R be a random variable, and let x be a positive real number. Then*

$$\Pr\{|R - \mathrm{E}[R]| \geq x\} \leq \frac{\mathrm{Var}[R]}{x^2}.$$

(Hint: Consider the case where $k = 2$).

Problem 4. Prove that the following two formulas for calculating variance are equivalent:

$$\mathrm{Var}[R] ::= \mathrm{E}\left[(R - \mathrm{E}[R])^2\right].$$
$$\mathrm{Var}[R] = \mathrm{E}\left[R^2\right] - \mathrm{E}^2[R],$$

A Appendix

The *expectation* of random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \mathrm{range}(R)} r \cdot \Pr\{R = r\}$$

The *variance*, $\mathrm{Var}[R]$, of a random variable, R, is:

$$\mathrm{Var}[R] ::= \mathrm{E}\left[(R - \mathrm{E}[R])^2\right].$$

Variance can also be equivalently defined as:

$$\mathrm{Var}[R] ::= \mathrm{E}\left[R^2\right] - \mathrm{E}^2[R],$$

Theorem (Markov's Theorem). *If R is a nonnegative random variable, then for all $x > 0$*

$$\Pr\{R \geq x\} \leq \frac{\mathrm{E}[R]}{x}.$$

Theorem (Chebyshev). *Let R be a random variable, and let x be a positive real number. Then*

$$\Pr\{|R - \mathrm{E}[R]| \geq x\} \leq \frac{\mathrm{Var}[R]}{x^2}.$$