Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 3, Wed

Problem 1. Let B be a set of numbers and let R be the divisibility relation on the set B. In other words, aRb iff $a \mid b$, that is, iff $ak = b$ for some natural number k.

(a) Let $B = \{2, 3, 4, 5, 6, 7, 8\}$. Draw a *Directed Acyclic Graph* (DAG) for the poset (B, R). What are the minimal and maximal elements? [1]

Solution. The minimal elements are 2, 3, 5, 7. The maximal elements are 5, 6, 7, 8.

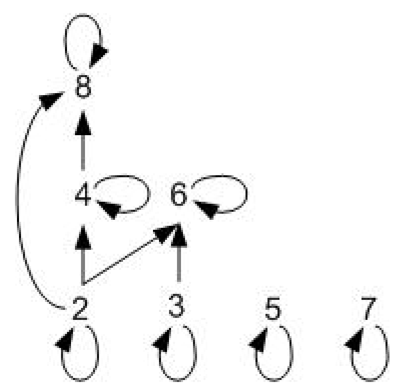

Figure 1: DAG for poset (B, R)

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(b) Let B be the set, $\mathbb{N}^+$, of integers greater than 1. What are the minimal and maximal elements?

Solution. All prime numbers are minimal elements of the poset (B, R) since no numbers divide them. Prime numbers divide all other numbers in their chains and can thus be considered the "head" of the chain.

There is no maximal element. For any n, we can create a larger number, namely $2n$, so the poset (B, R) does not have a maximal element. Note that a maximal element here is a number which is divisible by all the numbers in the chain (and can thus be considered as the "tail" of the chain). Since we can always make a larger number in the chain, no such maximal element exists. ■

[1] An element a is **maximal** in the poset $(S, \preceq)$ if there is no $b \in S$ such that $a \prec b$. Similarly, a is **minimal** if there is no element $b \in S$ such that $b \prec a$.

(c) Let B be the set, $\mathbb{N}$, of all natural numbers (including zero). What are the maximal and minimal elements?

Solution. The minimal element is 1 since 1 divides all natural numbers. The maximal element is 0 since all numbers divide 0. ■

Problem 2. For each student in our 6.042 class, we can assign a pair (s, a) where s represents the size (height) of the student and a represents the age of the student. Consider the relation, R, between pairs defined by the following condition:

$$(s_1, a_1)\, R\, (s_2, a_2) \quad \text{iff} \quad s_1 \leq s_2 \wedge a_1 \leq a_2.$$

At last count, our 6.042 class contained approximately 150 students. Prove that the class must contain either:

1. A set S of 13 students such that, if they line up according to increasing height, they are also arranged in increasing order of age, or
2. a set T of 13 students such that, if they line up according to increasing height, they are also arranged in decreasing order of age.

To eliminate some confusion: in these problems, when we say "increasing" we mean *weakly* increasing. A *weakly increasing* sequence may stay the same at some steps, it just never gets smaller; a *strictly increasing* sequence gets bigger at *every* step. For example, the sequence $\{2, 2, 3, 4\}$ is in *weakly* increasing order, but not in *strictly* increasing order.

Solution. For each student, we can assign a pair (s, a) where s represents the size (height) of the student and a represents the age of the student. We then get a set U of 150 such pairs. We can define the following binary relation R, denoted by $\preceq$, between two pairs P_1 and P_2: $P_1 \preceq P_2$ iff $s_1 \leq s_2$ and $a_1 \leq a_2$.

For example, if $P_1 = (60, 20)$, $P_2 = (60, 21)$, $P_3 = (65, 19)$, we have $P_1 \preceq P_2$ but P_3 and P_1 are incomparable.

It is easy to check that R is reflexive, transitive and antisymmetric. Thus, it defines a partial order on the set U. By Dilworth's Theorem, since U has 150 elements, it must either have a chain of size> 12 or an antichain of size $\geq 150/12$, that is, an antichain of size $\geq 25/2$.

Case 1 U has a chain of size > 12. Then it has a chain of size at least 12. This means that we can find 12 pairs which satisfy the relation $P_1 \preceq P_2 \preceq P_3 \ldots \preceq P_{12}$. From the definition of our binary relation ($\preceq$), this implies that $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_{12}$ AND $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_{12}$. Therefore, if we place the corresponding 12 students in the same order, they will be lined up in increasing order of size *and* in increasing order of age.

Case 2 U has an antichain of size $\geq 25/2$, so, since antichains only come in integer lengths, it has an antichain of size at least 13. Therefore, we can find 13 pairs $P_1, \ldots, P_{13}$, which are incomparable. We can line them up in order of increasing size (ignoring age), and come up with an order $P_1 \ldots P_{13}$ so that $s_i \leq s_{i+1} (i = 1 \ldots 13)$. Since for all i, $s_i \leq s_{i+1}$ and P_i, P_{i+1} are incomparable, we can conclude that $a_i > a_{i+1}$ (because if not, $a_i \leq a_{i+1}$ so for that i, $s_i \leq s_{i+1}$, $a_i \leq a_{i+1}$, therefore $P_i \preceq P_{i+1}$, so P_i and P_{i+1} would *not* be incomparable.) If we take the students corresponding to $P_1 \ldots P_{13}$ and line them up in that order, they will be in increasing order of size (since $s_i \leq s_{i+1}$) and in decreasing order of age (since we established that $a_i > a_{i+1}$). ■

Hint: Use:

Theorem (Dilworth). *For all $t \in \mathbb{N}$, every poset with n elements must have* either *a chain of size at least t*, or *an antichain of size at least n/t.*

Problem 3. A pair of 6.042 TAs, Adrian and Min, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Min's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Adrian's cat, Emilios - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.
7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

(a) Express the information in the task list using some type of graph (label the vertices to reflect task lengths).

Solution. The information in the table is represented in the Figure 2 in the form of a directed acyclic graph. Each vertex represents a task, and the weight of a vertex is the completion time. Each directed edge represents a dependency between tasks.

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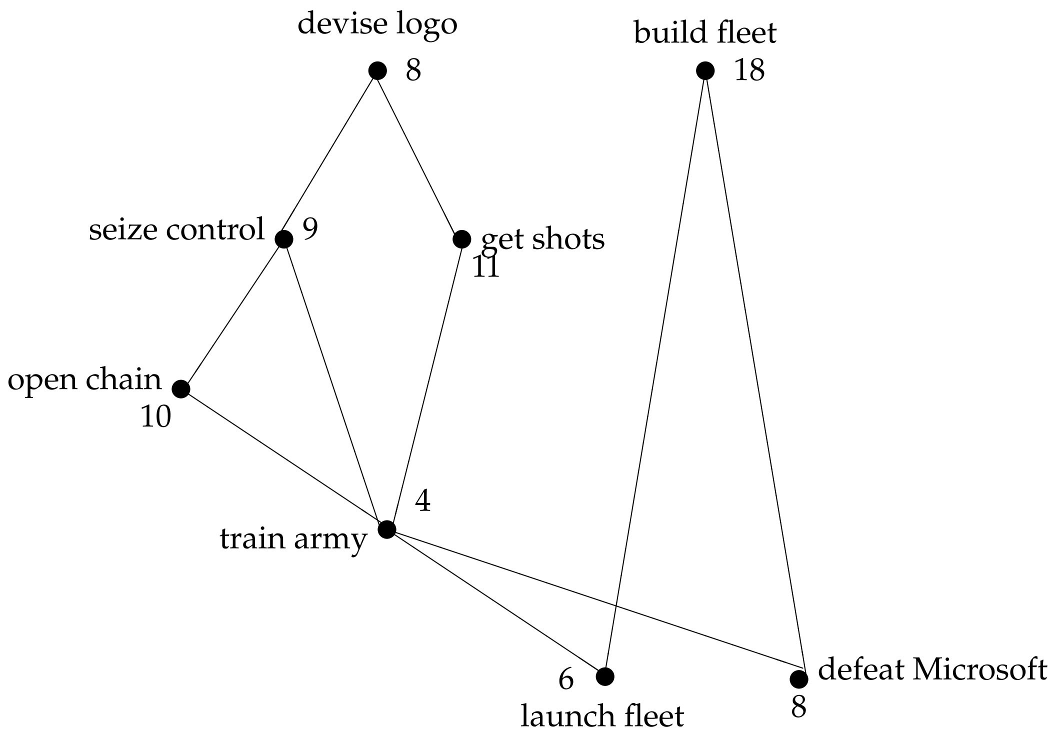

Figure 2: Graph representing the task precedence constraints.

(b) Give some valid order in which the tasks might be completed.

Solution. We can easily find several of them. The most natural one is valid, too: #1, #2, #3, #4, #5, #6, #7, #8. ■

Adrian and Min want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Adrian cannot work on building a fleet for a few days, run get shots for Emilios, and then return to building the fleet.

(c) Adrian and Min want to know how long conquering the galaxy will take. Min suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

Solution.

$$\frac{8+18+9+11+10+4+6+8}{2} = 37 \text{ days}$$

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Adrian and Min from both working all the time. ■

(d) Adrian proposes a different method for determining the duration of their project. He suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

Solution. The longest sequence of tasks is devising a logo (8 days), seizing the U. N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Adrian and Min may be unable to make progress on the critical path every day. ■

(e) What is the minimum number of days that Adrian and Min need to conquer the galaxy? No proof is required.

Solution. 40 days. Tasks could be divided as follows:

Min: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).

Adrian: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38). ■