Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 10, Wed

Problem 1. Suppose there is a system with n components, and we know from past experience that any particular component will fail in a given year with probability p. That is, letting F_i be the event that the ith component fails within one year, we have

$$\Pr\{F_i\} = p$$

for $1 \le i \le n$. The *system* will fail if *any one* of its components fails. What can we say about the probability that the system will fail within one year?

Let F be the event that the system fails within one year. Without any additional assumptions, we can't get an exact answer for $\Pr\{F\}$. However, we can give useful upper and lower bounds, namely,

$$p \le \Pr\{F\} \le np. \tag{1}$$

So for example, if $n = 100$ and $p = 10^{-5}$, we conclude that there is at most one chance in 1000 of system failure within a year and at least one chance in 100,000.

Let's model this situation with the sample space $\mathcal{S} ::= \mathcal{P}(\{1, \dots, n\})$ of subsets of positive integers $\le n$, where $s \in \mathcal{S}$ corresponds to the numbers of the components which fail within one year. For example, $\{2, 5\}$ is the outcome that the second and fifth components failed within a year and none of the other components failed. So the outcome that the system did not fail corresponds to the emptyset, $\emptyset$.

(a) Show that the probability that the system fails could be as small as p by describing appropriate probabilities for the sample points.

Solution. There could be a probability p of system failure if the all individual failures occur together. That is, let $\Pr\{\{1, \dots, n\}\} ::= p$, $\Pr\{\emptyset\} ::= 1 - p$, and let the probability of all other outcomes be zero. So $F_i = \{s \in \mathcal{S} \mid i \in s\}$ and $\Pr\{F_i\} = 0 + 0 + \cdots + 0 + \Pr\{\{1, \dots, n\}\} = \Pr\{\{1, \dots, n\}\} = p$. Also, the only outcome with positive probability in F is $\{1, \dots, n\}$, so $\Pr\{F\} = p$, as required. ■

(b) Show that the probability that the system fails could actually could be as large as np by describing appropriate probabilities for the sample points.

Solution. Suppose at most one component ever fails at a time. That is, $\Pr\{\{i\}\} = p$ for $1 \leq i \leq n$, $\Pr\{\emptyset\} = 1 - np$, and probability of all other points is zero. The sum of the probabilities of all the points is one, so this is a well-defined probability space. Also, the only sample point in F_i with positive probability is $\{i\}$, so $\Pr\{F_i\} = \Pr\{\{i\}\} = p$ as required. Finally, $\Pr\{F\} = np$ because $F = \{A \subseteq \{1, \dots, n\} \mid A \neq \emptyset\}$, so F in particular contains all the n outcomes of the form $\{i\}$. ■

(c) Prove the inequality (1).

Solution. $F = \bigcup_{i=1}^{n} F_i$ so

$$\begin{aligned}
p &= \Pr\{F_1\} && \text{(given)} && (2)\\
&\leq \Pr\left\{\bigcup F_i\right\} && \text{(monotonicity)} && (3)\\
&\leq \sum_{i=1}^{n} \Pr\{F_i\} && \text{(Boole's inequality)} && (4)\\
&= np. && && (5)
\end{aligned}$$

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(WE DIDN'T GET TO THE NEXT TWO PROBLEMS IN CLASS ON WEDNESDAY.)

Problem 2. Smith and Wesson are shooting at a target. Suppose Smith's chance of hitting the target is double that of his friend Wesson, and the probability that at least one of them hits the target is $1/2$. Whether or not one of them hits the target has no effect on the probability that the other one will hit. What is the probability that Wesson hits the target?

Solution. A simple sample space has length two sequences of *H*its and *M*iss indicating whether Smith hit the target and then Wesson did:

$$\mathcal{S} ::= \{HH, HM, MH, MM\}.$$

Let S be the event that Smith hits, that is

$$S ::= \{HH, HM\}.$$

Similarly, let W be the even that Wesson hits:

$$W ::= \{HH, MH\}.$$

First we have two facts:

$$\begin{aligned}
\Pr\{S\} &= 2\Pr\{W\} && (6)\\
\Pr\{S \cup W\} &= \frac{1}{2}, && (7)
\end{aligned}$$

To define the probability space, we would draw a binary tree assigning probability $\Pr\{S\} \cdot \Pr\{W\}$ to the event $\{HH\} = S \cap W$ and also $\Pr\{\overline{S}\} \cdot \Pr\{W\}$ to $\{MH\}$. That is, the probability that Smith hits the target and *then* Wesson does is the probability that Smith hits the target *times* the probability that Wesson does.

$$\Pr\{S \cap W\} = \Pr\{S\} \Pr\{W\} \tag{8}$$

We should verify that in this probability space, event S has the required probability. Namely,

$$\begin{aligned}
\Pr\{S\} &::= \Pr\{\{HH, HM\}\} && \text{(def of } S\text{)}\\
&= \Pr\{S\}\Pr\{W\} + \Pr\{S\}\Pr\{\overline{W}\}, && \text{(probabilities from the tree)}\\
&= \Pr\{S\}\Pr\{W\} + \Pr\{S\}(1 - \Pr\{W\}) && \text{(Complement rule)}\\
&= \Pr\{S\}(\Pr\{W\} + (1 - \Pr\{W\})) && \text{(distributivity)}\\
&= \Pr\{S\},
\end{aligned}$$

and similarly for $\Pr\{W\}$.

Expanding equation (7) using inclusion-exclusion gives:

$$\begin{aligned}
\frac{1}{2} &= \Pr\{S \cup W\} && \text{by (7)} && (9)\\
&= \Pr\{S\} + \Pr\{W\} - \Pr\{S \cap W\} && \text{(inclusion-exclusion)} && (10)\\
&= \Pr\{S\} + \Pr\{W\} - \Pr\{S\} \cdot \Pr\{W\} && \text{(by (8))} && (11)\\
&= 2\Pr\{W\} + \Pr\{W\} - 2\Pr\{W\} \cdot \Pr\{W\} && \text{by (6).} && (12)
\end{aligned}$$

Hence,

$$3\Pr\{W\} - 2\Pr\{W\}^2 - \frac{1}{2} = 0. \tag{13}$$

Equation (13) is a quadratic in $\Pr\{W\}$; only one of its roots lies between 0 and 1 and so makes sense as a probability, namely,

$$\frac{3}{4} - \frac{1}{4}\sqrt{5} \approx 0.19.$$

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Problem 3. Carried over to in-class Problem 1 Friday, Week 10.