Massachusetts Institute of Technology
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Solutions to In-Class Problems — Week 7, Fri

Problem 1. There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Solution. $a = -1$.

For $p = -1$, the sum is the harmonic series which we know does not converge. Since the term i^p is increasing in p for $i > 1$, the sum will be larger, and hence also diverge for $p > -1$.

By the integral method, the sum is Θ of the integral from 1 to ∞ of x^p. For $p < -1$, the indefinite integral is $x^{p+1}/(p+1) = \Theta(1/x^{\epsilon})$ for $\epsilon = -1-p > 0$, so the integral evaluates to a constant. Hence the sum is bounded above, and since it is increasing, it has a finite limit, *i.e.*, it converges. ■

Problem 2. Suppose $f_1, f_2, g_1, g_2 : \mathbb{N} \to \mathbb{N}$. Prove the following:

(a) If $f_1 = o(g_1)$, then $f_1 + g_1 \sim g_1$.

Solution.

$$\lim_{n\to\infty} \frac{f_1 + g_1}{g_1} = \lim_{n\to\infty} \frac{f_1}{g_1} + \frac{g_1}{g_1} = 0 + 1 = 1.$$

■

(b) If $f_1 = o(g_1)$ and $f_2 = o(g_2)$, then $f_1 + f_2 = o(g_1 + g_2)$.

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \frac{f_1 + f_2}{g_1 + g_2} &= \lim_{n\to\infty} \frac{f_1 + f_2}{g_1 + g_2} \\
&= \lim_{n\to\infty} \frac{f_1}{g_1 + g_2} + \frac{f_2}{g_1 + g_2} \\
&\leq \lim_{n\to\infty} \frac{f_1}{g_1} + \frac{f_2}{g_2} \\
&= \lim_{n\to\infty} \frac{f_1}{g_1} + \lim_{n\to\infty} \frac{f_2}{g_2} \\
&= 0 + 0 = 0.
\end{aligned}$$

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(c) If $f_1 = O(g_1)$ and $f_2 = O(g_2)$, then $f_1 + f_2 = O(g_1 + g_2)$.

Solution. Say $f_1(n) \leq c_1 g_1(n)$ and $f_2(n) \leq c_2 g_2(n)$ for some constants $c_1, c_2 \geq 0$ and all large n. Then

$$f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n) \leq c(g_1(n) + g_2(n)),$$

where $c = \max c_1, c_2$. ■

(d) If $f_1 \sim g_1$ and $f_1 \geq f_2 \sim g_2 \leq g_1$, then it is *not* necessarily true that $f_1 - f_2 \sim g_1 - g_2$.

Solution. Let $f_1 ::= n + 1, g_1 ::= n + 2, f_2 ::= g_2 ::= n$. ■

Problem 3. Indicate which of the following holds for each pair of functions $(f(n), g(n))$ in the table below. Assume $k \geq 1$, $\epsilon > 0$, and $c > 1$ are constants.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
2^n	$2^{n/2}$						
$\sqrt{n}$	$n^{\sin n\pi/2}$						
$\log(n!)$	$\log(n^n)$						
n^k	c^n						
$\log^k n$	n^ϵ						

Solution.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
2^n	$2^{n/2}$	no	no	yes	yes	no	no
$\sqrt{n}$	$n^{\sin n\pi/2}$	no	no	no	no	no	no
$\log(n!)$	$\log(n^n)$	yes	no	yes	no	yes	yes
n^k	c^n	yes	yes	no	no	no	no
$\log^k n$	n^ϵ	yes	yes	no	no	no	no

Following are some hints on deriving the table above:

(a) $\dfrac{2^n}{2^{n/2}} = 2^{n/2}$ grows without bound as n grows—it is not bounded by a constant.

(b) When n is even, then $n^{\sin n\pi/2} = 1$. So, no constant times $n^{\sin n\pi/2}$ will be an upper bound on $\sqrt{n}$ as n ranges over even numbers. When $n \equiv 1 \bmod 4$, then $n^{\sin n\pi/2} = n^1 = n$. So, no constant times $\sqrt{n}$ will be an upper bound on $n^{\sin n\pi/2}$ as n ranges over numbers $\equiv 1 \bmod 4$.

(c)

$$\log(n!) = \log \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \pm c_n \quad (1)$$
$$= \log n + n(\log n - 1) \pm d_n \quad (2)$$
$$\sim n \log n \quad (3)$$
$$= \log n^n.$$

where $a \leq c_n, d_n \leq b$ for some constants $a, b \in \mathbb{R}$ and all n. Here equation (1) follows by taking logs of Stirling's formula, (2) follows from the fact that the log of a product is the sum of the logs, and (3) follows from Problem 2, part a because any constant, $\log n$, and n are all $o(n \log n)$.

(d) *Polynomial* growth versus *exponential* growth.

(e) *Polylogarithmic* growth versus *polynomial* growth.

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Problem 4. It is a standard fallacy to think that given n quantities each of which is $O(1)$, their sum would have to be $O(n)$. In fact, such a sum can grow arbitrarily fast:

Let $g : \mathbb{N} \to \mathbb{N}^+$ be *any* function. Explain how to define a sequence $f_1, f_2, \ldots$ of functions from $\mathbb{N}$ to $\mathbb{N}$ such that

$$f_i(n) = O(1) \quad (4)$$

for all $i \geq 1$, but

$$\sum_{i=1}^{n} f_i(n) \neq O(g(n)). \quad (5)$$

Solution. Pick f_i to be the constant function $ig(i)$. That is,

$$f_i(n) ::= ig(i),$$

for all $i \geq 1$ and $n \geq 0$. Since f_i is a constant function, it is $O(1)$, that is, condition (4) holds.

But

$$\sum_{i=1}^{n} f_i(n) \geq f_n(n) = ng(n),$$

and $ng(n) \neq O(g(n))$, so (5) also holds. In fact, since $g(n) = o(ng(n))$, we have the stronger condition that

$$g(n) = o\left(\sum_{i=1}^{n} f_i(n)\right).$$

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