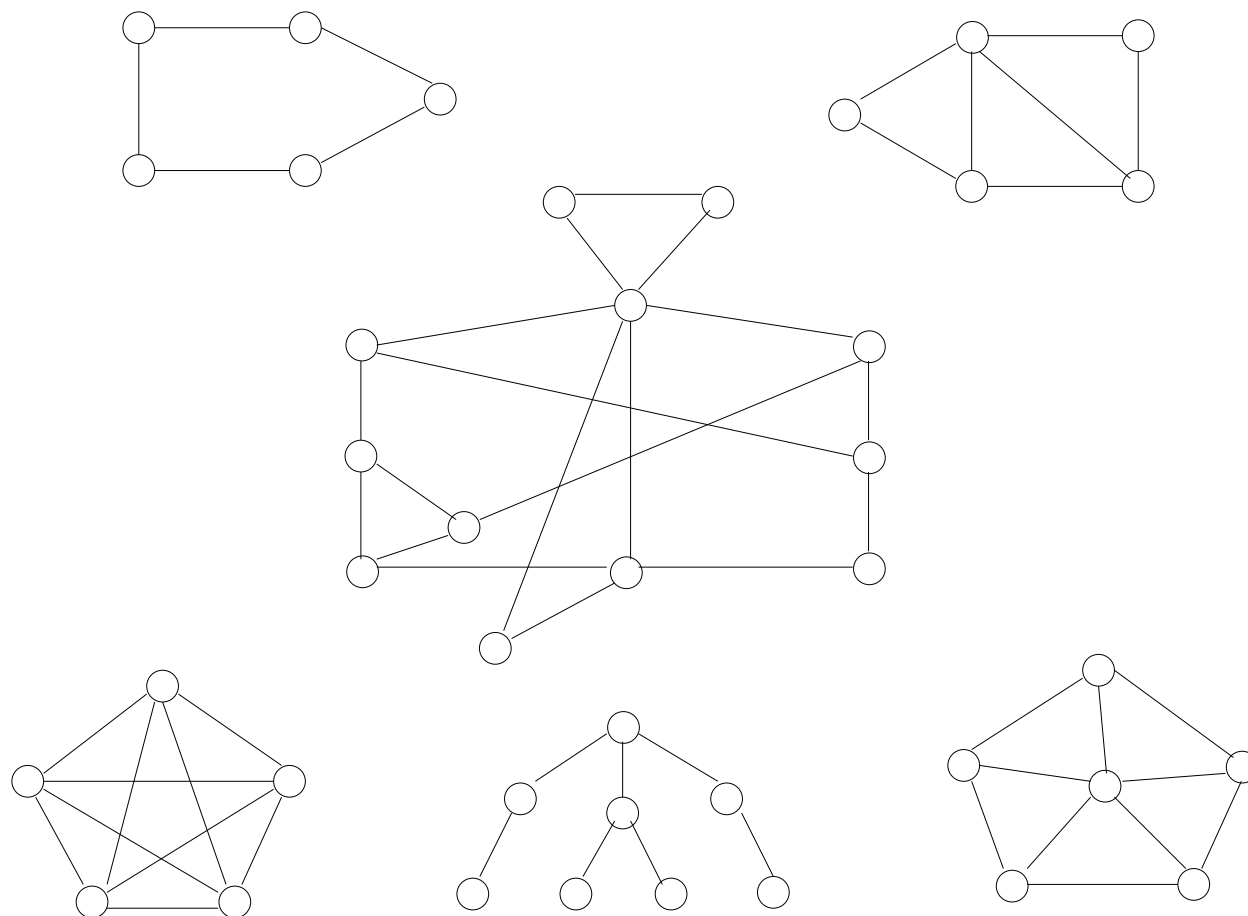


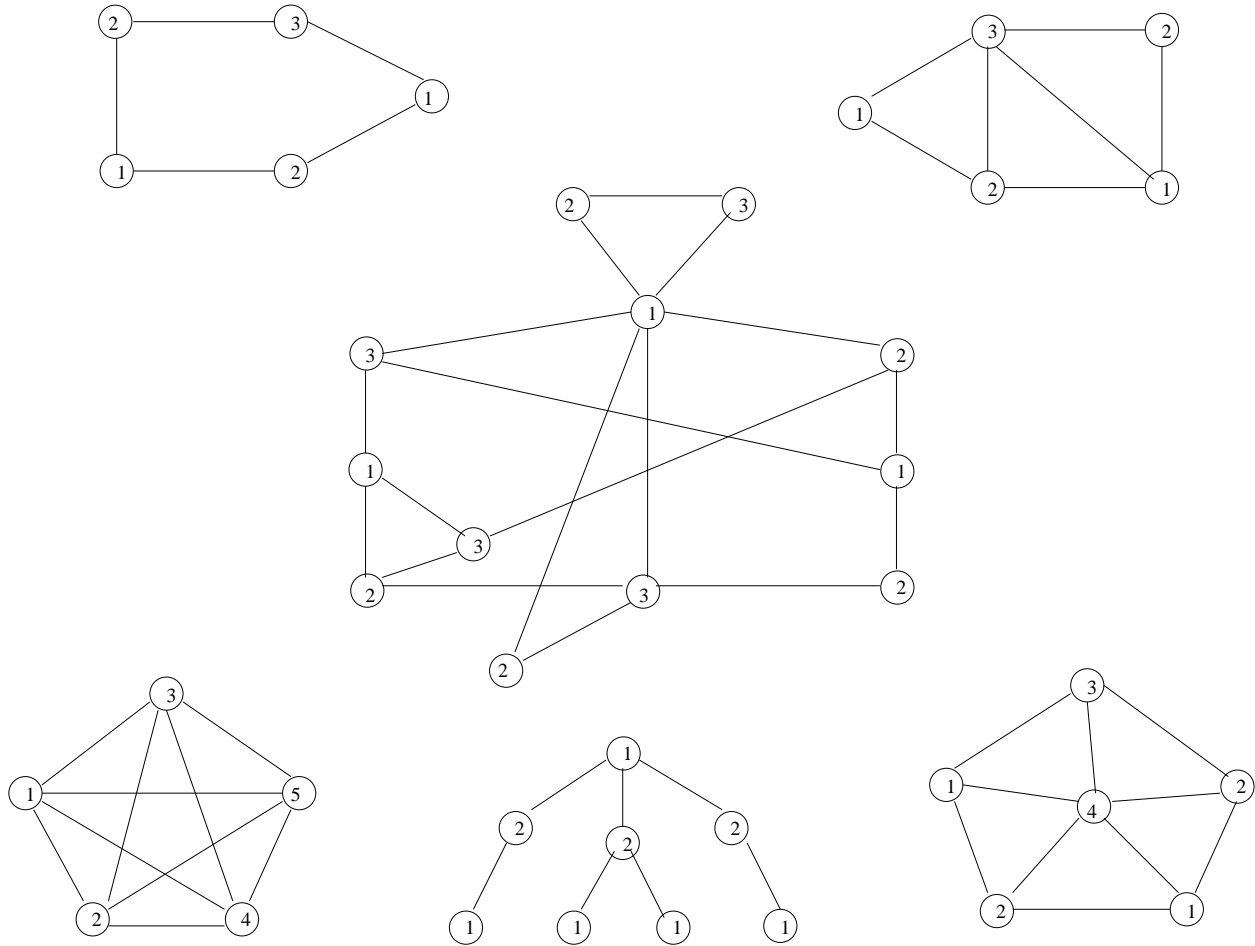
## Solutions to In-Class Problems — Week 4, Wed

**Problem 1.** Color the following graphs with the minimum possible colors, such no two adjacent vertices have the same color.



**Solution.**

■



*Definition:* The degree of a vertex is the number of edges connecting it to other vertices.

*Theorem:* The sum of the degrees of the vertices in a simple graph equals twice the number of edges, i.e.  $2|E| = \sum_{v \in V} \text{deg}(v)$ .

*Handshaking Theorem:* In every graph, there are an even number of vertices of odd degree.

**Problem 2.** Prove that in any undirected graph, the sum of the squares of the vertex degrees is even, or provide a counterexample.

**Solution.** The assertion is true.

*Proof 1*

Let  $V_e$  be the set of even degree vertices and  $V_o$  the set of odd degree vertices of the graph. From class we know that there is an even number of odd degree vertices. So can write

$$\sum_{v \in V} \deg(v)^2 = \sum_{v \in V_e} \deg(v)^2 + \sum_{v \in V_o} \deg(v)^2 \quad (1)$$

If  $v \in V_e$  then  $\deg(v)$  is even, hence  $\deg(v)^2$  is also even. Therefore, the first term in the right hand side of (1) is even. Similarly if  $v \in V_o$ , then  $\deg(v)^2$  is odd. The second term in the right hand side is a sum of odd numbers with an even number of terms. Therefore it is also even. We conclude that  $\sum_{v \in V} \deg(v)^2$  is even

*Proof 2*

Let  $E$  be the set of edges of the graph and  $V = \{v_1, \dots, v_n\}$  be the set of vertices of the graph.

$$\left(\sum_{i=1}^n \deg(v_i)\right)^2 = \sum_{i=1}^n \deg(v_i)^2 + 2 \sum_{i \neq j} \deg(v_i) \deg(v_j) \quad (2)$$

The left hand side of (2) is even because  $\sum_{i=1}^n \deg(v_i) = 2|E|$  and the second part of the right hand side is obviously even. Therefore  $\sum_{i=1}^n \deg(v_i)^2$  is even. ■

**Problem 3.** Prove that it is possible to color a graph  $G$  with  $dmax + 1$  colors, where  $dmax$  represents the maximum degree of any node in  $G$ .

**Solution.** There are many different ways of proving this claim. Here we'll use induction on the size of the graph. This approach also yields a recursive algorithm for coloring the graph.

*Proof.* Let  $P(n)$  be the predicate that it is possible to color every  $n$ -vertex graph  $G$  with at most  $dmax + 1$  colors.

**Base case:** Consider a graph  $G$  with 1 vertex ( $n = 1$ ). This vertex has degree zero, so  $dmax = 0$ . Since we can color that lone vertex with  $dmax + 1 = 1$  color,  $P(1)$  holds.

**Inductive step:** we assume  $P(n)$  to prove  $P(n + 1)$ . Consider an  $(n + 1)$ -vertex graph  $G$  with maximum degree  $dmax$ . Let  $G'$  be the graph obtained from  $G$  by (temporarily) removing an arbitrary vertex  $v$  and all incident edges and let  $dmax'$  be its maximum degree. Since  $G'$  now has  $n$  vertices, we can apply the inductive hypothesis which says that we can color it with at most  $dmax' + 1$  colors.

We now add vertex  $v$  to get back  $G$ . This cannot *decrease* the maximum degree, so  $dmax \geq dmax'$ . Therefore,  $dmax + 1$  colors suffice to color all the nodes other than  $v$  (that is,  $G'$ ). As for  $v$ , since it can have degree at most  $dmax$ , it has at most  $dmax$  neighbors and we have  $dmax + 1$  colors at our disposal, so we can always pick one which is not used by any of its neighbors.

This proves that  $G$  is also colorable, hence  $P(n + 1)$  is true.

□

Note that this theorem implies, for example, that a graph with thousands of vertices, each of degree 3, requires at most 4 colors. So for example the whole US worth of wireless devices would only require a number of distinct frequencies comparable to the local radius (neighborhood) of a single wireless device (as opposed to the number of total devices). ■