Mathematics for Computer Science

MIT 6.042J/18.062J

Great Expectations

Prediction is difficult, especially of the future

--- *Niels Bohr*

Abstractions in Probability

- Random Variables $R(s) = r$
- Probability Density Function
  $\Pr\{R = r\}$ = for all outcomes s such that $R(s) = r$, sum the probability of s

Expected Value

- Fortune Telling - Average of *future* values

Average vs Expected Value

R = number of heads in 2 coin flips

Average of 10 experiments:

= 1 (head) 5/10 + 2 (heads) 3/10

Expected Value:

= 1 . Pr{1 head} + 2. Pr {2 heads}

Definition

Expected value

of a random variable R

$$E[R] ::= \sum_{r \in range(R)} r \cdot \Pr\{R = r\}$$

A Simple Example

R = value thrown by a dice

- Expected value of *R*

$$= \sum_{r \in range(R)} r \cdot \Pr\{R = r\}$$

= 1(1/6) + 2(1/6) + 3 (1/6) + 4(1/6) + 5(1/6) + 6(1/6)

= 3.5

Carnival Dice

- Player chooses a number from 1 to 6
- Roll 3 fair and independent Dice
  - Win $1 if *any* match
  - Lose $1 if *no* match

Carnival Dice

- R ::= Profit on a Roll
- R can take on values 1 or -1

- E[R] = -1 Pr {no match} + 1 Pr{any match}

  $= -1.\ (5/6)^3 + 1.(1 - (5/6)^3)$

  = - 34/216

Expect to *Lose 16 cents* on average

Modified Carnival Dice

- Player chooses a number from 1 to 6
- Roll 3 fair and independent Dice
  - Win $1 for *each* match
  - Lose $1 if *no* match

In-class Problem

Problem 1

Other Applications

- Gambling
  - Will the Patriots win the Superbowl?
  - How many rounds of 5 card draw do I expect to play before I finally win?
- Algorithms/System Design
  - Given a sequence of *n* numbers, on average how many numbers are out of order?
- Models
  - Biology, physics, economy, voting….

Pitfalls of Expectation

Expected Value ≠ value you expect to see

- Same Expectation for different distributions
- Expected value may never actually occur!

Model is not Reality

Birthday Problem

- In a class with 150 people, what is the probability that two people have the same birthday?
  - 99%
- How many pairs of people with the same birthday do we expect to see?

Real Data (for this Class)

- 17 pairs and 2 triples = 23 pairs
- Class size (approx 150 students)
- Should you expect to see something similar in other classes?

Birthday Problem

R ::= # pairs with same birthday

- Pr{$R = 0$}
- Pr{exactly one pair: R=1}
- Pr{exactly two pairs: R=2} = ?

Starts becoming very messy …

Linearity of Expectation

$$E[R_1 + R_2 + \ldots + R_k] = E[R_1] + E[R_2] + \ldots + E[R_k]$$

Birthdays

Let $R_i = 1$ if i^th pair has same birthday,
$= 0$ if not

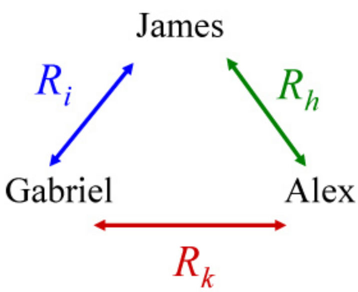
Pr{$R_i = 1$} = 1/365 = p

Pr{$R_i = 0$} = $1 - p$

$E[R_i] = 1.p + 0\,(1-p) = p$

R_i is an Indicator Random Variable

Note: R_i are Not Independent



Expected Number of Birthday Pairs

E[R]

$= E[R_1 + R_2 + R_3 \ldots.]$

$= E[R_1] + E[R_2] + E[R_3] \ldots.$

$= \binom{n}{2} \cdot E[R_i]$

$= N(N\text{-}1) / 2 \,.\, 365$

~ 30 pairs for 6.042

The Same Question

N wireless devices randomly pick names from a range R.

- How many name collisions do you expect to see?

  $= N^2 / 2R$

The Hat Check Game

Everyone puts their name in a hat, then each person randomly draws out a name.

How many people expect to get their own name back?

Collect stats per table

In-class Problems

Problems 2,3,4