Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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In-Class Problems — Week 2, Fri

Problem 1. What amounts of postage can be made from 6¢ stamps and 10¢ stamps? Prove it.

Problem 2. Pinpoint, and illustrate with a counterexample, *exactly* where the following proof goes awry.

An integer, m, *divides* an integer, n, in symbols, $m \mid n$, iff there is an integer k such that $km = n$.

False Claim. *For any positive integers $p, x_1, x_2, \ldots, x_n$, if $p \mid x_1 x_2 \cdots x_n$, then $p \mid x_i$ for some i between 1 and n.*

False proof. (by strong induction on n.)

The induction hypothesis is the False Claim itself.

Base case ($n = 1$): When $n = 1$, we have $p \mid x_1$, therefore we can let $i = 1$ and conclude $p \mid x_i$.

Induction step: Now assuming the claim holds for all $k \leq n$, we must prove it for $n + 1$.

So suppose $p \mid x_1 x_2 \cdots x_{n+1}$. Let $y_n = x_n x_{n+1}$, so $x_1 x_2 \cdots x_{n+1} = x_1 x_2 \cdots x_{n-1} y_n$. Since the righthand side of this equality is a product of n terms, we have by induction that p divides one of them. If $p \mid x_i$ for some $i < n$, then we have the desired i. Otherwise $p \mid y_n$. But since y_n is a product of the two terms x_n, x_{n+1}, we have by strong induction that p divides one of them. So in this case $p \mid x_i$ for $i = n$ or $i = n + 1$. □

Problem 3. Here's a different proof that the square root of 2 is irrational, this one based on the Least Number Principle.

Proof. The proof is by contradiction.

Assume that the square root of 2 is rational. Then $\sqrt{2}$ is equal to m/n for some positive integers m and n. So n has the property that

$$n \text{ and } n\sqrt{2} \text{ are positive integers.} \tag{1}$$

Now by the Least Number Principle, there must be a *smallest* positive integer, n, satisfying (1). For this n, let $n_0 ::= n(\sqrt{2} - 1)$. But then n_0 is smaller than n, and n_0 and $n_0\sqrt{2}$ are positive integers, a contradiction. □

(a) Are you convinced by this proof? Briefly explain.

In particular, why is n_0 an integer?

Why is n_0 positive?

Why is n_0 smaller than n?

Why is $n_0\sqrt{2}$ a positive integer?

We can generalize this proof to prove that $\sqrt{k}$ is irrational for integers k other than 2. Simply revise the next to last sentence to read, " ... and let $n_0 ::= n(\sqrt{k} - \left\lfloor\sqrt{k}\right\rfloor)$". (For any real number, r, the expression $\lfloor r \rfloor$ denotes r rounded *down* to the nearest integer.)

(b) Are you convinced by this argument? Explain.

Problem 4. We are given a chocolate bar with $m \times n$ squares of chocolate, and our task is to divide it into mn individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break.

For example, suppose that the chocolate bar is 2×2. The first split makes two pieces, both 2×1. Each of these pieces requires one more split to form single squares. This gives a total of three splits.

(a) Use strong induction to conclude the following:

Theorem. *To divide up a chocolate bar with $m \times n$ squares, we need at most $mn - 1$ splits.*

(b) The theorem proves that *at most* $mn - 1$ splits are needed to divide up an m by n chocolate bar. Could you do better?