Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Prof. Albert Meyer and Dr. Radhika Nagpal

In-Class Problems — Week 12, Fri

Problem 1. Consider the excerpt from Herbert Simon's paper "The Architecture of Complexity" Proceedings of the American Philosophical Society, vol 106, no 6, Dec 1962, page 470.

(a) What is wrong with the analysis?

(b) What is the correct expected time for Hora and Tempus to complete a watch?

Problem 2. The St. Petersburg Casino offers the following game: the gambler bets a fixed wager, and then the dealer flips a fair coin (dealers do not flip coins in US casinos, but they do in St. Petersburg) until it comes up heads. The gambler receives \$1 if the coin shows heads the first time, \$2 if it shows the first head at second toss, and in general \$ 2^{k-1} if the dealer tosses the coin k times to get the first head.

(a) Suppose the fixed wager is \$10. What is the expected amount of money that the gambler will win in this game? Suppose the fixed wager is \$10,000?

(b) What is the probability that the gambler does not lose money in a game when the fixed wager is \$10,000?

(c) In reality, it would not be reasonable for the gambler to play the game with the fixed wager at \$10,000. Why? (*Hint:* Suppose the casino has a limit of a billion dollars.)

Problem 3. Just like event probabilities, expectations can be conditioned on some event. We define *conditional expectation*, $\mathrm{E}[R \mid A]$, *of a random variable*, R, *given event*, A:

$$\mathrm{E}[R \mid A] ::= \sum_r r \cdot \Pr\{R = r \mid A\}. \tag{1}$$

In other words, it is the expected value of the variable R once we skew the distribution of R to be conditioned on event A. A real benefit of conditional expectation is the way it lets us divide complicated expectation calculations into simpler cases.

Theorem 3.1. *[Law of Total Expectation] If the sample space is the disjoint union of events $A_1, A_2, \cdots$, then*

$$\mathrm{E}[R] = \sum_i \mathrm{E}[R \mid A_i] \Pr\{A_i\}.$$

Prove the above theorem.

Problem 4. Compute the expected value for each of the following random variables. (Assume that all dice are fair and six-sided and that dice rolls are mutually independent.)

(a) The sum of the rolls of three dice.

(b) The product of the rolls of three dice.

(c) The sum of the rolls of a number of dice that is given by the roll of a single die. (For example, if you roll a 3 on the single die, then you take the sum of 3 dice rolls; if you roll a 5 on the single die, then you take the sum of 5 dice rolls.)

(d) Suppose now that the dice rolls are not guaranteed to be mutually independent. Which of your answers above must still be correct?

A Appendix

The *expectation* of random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}.$$

If R has codomain $\mathbb{N}$, then this definition can also be written as

$$\mathrm{E}[R] = \sum_{r \in \mathbb{N}} \Pr\{R > r\}$$

Theorem. *Let $C_1, C_2, \ldots,$ be a sequence of nonnegative random variables, and let Q be a positive integer-valued random variable, all with finite expectations. Suppose that*

$$\mathrm{E}[C_i \mid Q \geq i] = \mu$$

for some $\mu \in \mathbb{R}$ and for all $i \geq 1$. Then

$$\mathrm{E}[C_1 + C_2 + \cdots + C_Q] = \mu \, \mathrm{E}[Q].$$