Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

# Problem Set 2

**Reading:** Week 2 Notes, Rosen §3.2, 3.3 through Example 6.

**Reading Comments (required):** By 11am on the *Reading Problems due date* (which is generally the week before the pset due date), send email to

indicating in at most three sentences one or more of the following: the passage in the reading—including its page number—you

- found most difficult,
- found most surprising,
- would like to have discussed in the next lecture.

**Problem 1.** Use induction to prove that the following equation holds for all $n \geq 2$.

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{n}\right)=\frac{1}{n}$$

**Problem 2.** Suppose you take a piece of paper and draw a bunch of straight lines, no one exactly on top of another, that completely cross the paper. This divides the paper up into polygonal regions. Prove by induction that you can always color the various regions using only *two* colors, so that any two regions that share a boundary line are different colors. Regions that share only a boundary point may have the same color.

**Problem 3.** The 6.042 course information sheet states:

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> Tutorials will be devoted to solving interesting problems, with students working in teams of three or four.

Using strong induction, prove that if a recitation contains at least 6 students, then the class can be divided into teams, each consisting of either 3 or 4 students.

**Problem 4.** Let the function, $g$, be defined on the natural numbers recursively as follows: $g(0) = 0$, $g(1) = 1$, and $g(n) = 5g(n-1) - 6g(n-2)$, for $n \geq 2$.

Prove that for all $n \in \mathbb{N}$, $g(n) = 3^n - 2^n$.

**Problem 5.** Ackermann's function, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, satisfies the following equations:

$$
\begin{aligned}
f(m,n) &= 2n, && \text{if } m = 0 \text{ or } n \leq 1, && (1)\\
f(m,n) &= f(m-1, f(m, n-1)), && \text{if } m > 0 \text{ and } n > 1. && (2)
\end{aligned}
$$

These equations actually determine Ackermann's function uniquely. That is,

**Lemma.** *If $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ also satisfies the same equations as $f$, namely,*

$$
\begin{aligned}
g(m,n) &= 2n, && \text{if } m = 0 \text{ or } n \leq 1, && (3)\\
g(m,n) &= g(m-1, g(m, n-1)), && \text{if } m > 0 \text{ and } n > 1, && (4)
\end{aligned}
$$

*then $f = g$.*

**(a)** Prove this Lemma *by induction*. *Hint:* Try the induction hypothesis

$$P(m) ::= \forall n \; f(m,n) = g(m,n).$$

**(b)** Now prove this Lemma *using the Least Number Principle*. *Hint:* Consider the least $m$ such that $\exists n \; f(m,n) \neq g(m,n)$.

**Problem 6.** An *$n$-player tournament* consists of some set of $n \geq 2$ *players*, and, for every two distinct players, a specification that one of the players beats the other. That is, player $p$ beats player $q$ iff $q$ does not beat $p$, for all players $p \neq q$.

A sequence of distinct players $p_1, p_2, \ldots, p_k$, such that player $p_i$ beats player $p_{i+1}$ for $1 \leq i < k$ is called a *ranking* of these players. If also player $p_k$ beats player $p_1$, the ranking is called a *$k$-cycle*.

**(a)** Prove by induction that in every tournament, either there is a "champion" player that beats every other player, or there is a 3-cycle.

**(b)** A *consistent ranking* is a sequence $p_1, p_2, \ldots, p_n$ of all $n$ players in the tournament such that $p_i$ beats $p_j$ iff $i < j$, for $1 \leq i, j \leq n$. Conclude that a tournament has no consistent ranking iff some subset of three of its players has no consistent ranking.