In-Class Problems — Week 12, Mon

Problem 1. In the original game of Carnival Dice, the player chooses a number from 1 to 6. She then throws three fair and mutually independent dice. She wins one dollar if any die matches, and loses a dollar otherwise. This is a losing proposition for the player.

Consider a modified version of Carnival Dice. The game is the same except the player wins one dollar for *each* die that matches her number, and she loses one dollar if no die matches. Is this a good game to play? What is her expected profit?

Problem 2. There is a dinner party where N people check their hats. The hats are mixed up during dinner, so that afterward each person receives a random hat. In particular, each person gets their own hat with probability $1/N$. What is the expected number of people who get their own hat?

Problem 3. Prove that $\mathrm{E}[R_1 + R_2] = \mathrm{E}[R_1] + \mathrm{E}[R_2]$, where R_1 and R_2 are two random variables defined on the same sample space, but not necessarily independent. (*Hint:* Start from Definition 4.1).

Problem 4. Let R be the number of heads that come up when we toss n independent coins, where each coin comes up heads with probability p. The random variable R has a binomial distribution. Prove that the expected value of R is np.

A Appendix

Definition 4.1. The *expectation*, $\mathrm{E}[R]$, of a random variable, R, on the sample space S, is defined as:

$$\mathrm{E}[R] ::= \sum_{s \in \mathcal{S}} R(s) \cdot \Pr\{s\}. \tag{1}$$

Another equivalent definition is:

Definition 4.2. The *expectation* of random variable, R, is:

$$\mathrm{E}[R] ::= \sum_{r \in \text{range}(R)} r \cdot \Pr\{R = r\}. \tag{2}$$

Theorem 4.3. *If R is a random variable with range $\mathbb{N}$, then*

$$\mathrm{E}[R] = \sum_{i=0}^{\infty} \Pr\{R > i\}.$$

Theorem 4.4. *(Expectation of a Sum) For any random variables $R_1, \ldots, R_k$,*

$$\mathrm{E}\left[\sum_{i=1}^{k} R_i\right] = \sum_{i=1}^{k} \mathrm{E}[R_i].$$

Theorem 4.5. *If $|x| < 1$, then*

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}.$$