*Mathematics for Computer Science*

**MIT 6.042J/18.062J**

# Combinatorics II

## Last Week: Counting I

- Sets
  - Bijections, Sum Rule, Product Rule, Inclusion-Exclusion
- Pigeonhole Principle
- Permutations
- Tree Diagrams

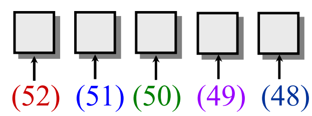
## This Week: Counting II

- Division Rule
- Combinations
  *(binomial coefficients)*
- Binomial Theorem and Identities
- Permutations with Repetition
  *(multinomial coefficients)*
- Combinations with Repetition
  *(stars and bars)*

## The Real Agenda: Poker

How many different 5-card poker hands are there?

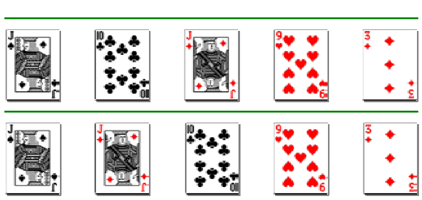



This method counts the number $P(52,5)$ of ***5-permutations*** from a set of 52 elements $= 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 311{,}875{,}200$.

## Problem: Overcounting

These two hands are the same:



In fact, *any permutation* of these cards is the same hand. The order of cards is irrelevant.

## Division Rule

If set $B$ counts every element of set $A$ exactly $k$ times, then

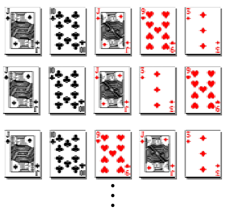
$$|A| = |B|/k$$

## Number of 5-Card Hands

How much have we overcounted?

☞ Every hand is counted 5! times.



***Division rule:***

$$\#\,\text{hands} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2{,}598{,}960$$




## Combinations

$C(n, r)$ = number of different size-$r$ subsets of a size-$n$ set.

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{0} = \binom{n}{n} = 1$$

$$\binom{n}{1} = \binom{n}{n-1} = n$$




## Poker: Ranks of Hands

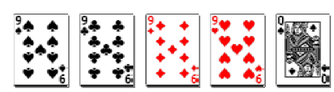
Straight Flush
- > Four-of-a-Kind
  - > Full House
    - > Flush
      - > Straight
        - > Three-of-a-Kind
          - > Two Pairs
            - > One Pair
              - > High Card




## Four-of-a-Kind

- 4 cards with the same value
- 1 card with a different value

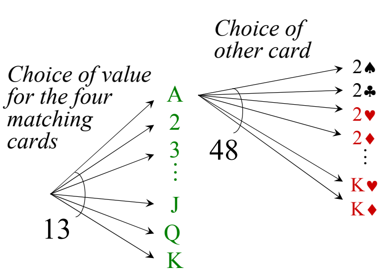
***Example:***



Each card has one of 13 ***values*** and one of 4 ***suits***.



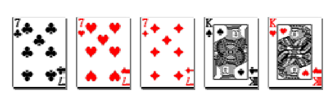

## Counting Fours-of-a-Kind




# Fours-of-a-Kind = 13 · 48 = 624




## Full House



# full houses = # choices of value for the triple
× # choices for their 3 suits
× # choices of value for the pair
× # choices for their 2 suits

$$= \binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2}$$

$$= 13 \cdot 4 \cdot 12 \cdot 6$$

$$= 3744$$

## In-Class Problem 1

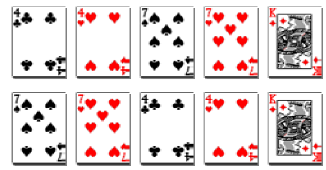
Please write your solution
on the whiteboard.




## Incorrect Counting Argument

$$\#\text{ two-pairs} = \binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}44$$

Every hand is counted twice!






## Correct Counting Arguments

1. Divide the Theory Hippo's count by 2.

2. Choose the values of the two pairs together:

$$\#\text{ two-pairs} = \binom{13}{2}\binom{4}{2}\binom{4}{2}44$$




## Permutations vs. Combinations

***Combinations:***
- Subsets of size $r$.
- Order of elements *does not* matter.
- $C(n, r)$: counting all $r$-permutations overcounts every combination by $r!$.

***Permutations:***
- Strings of length $r$.
- Order of elements *does* matter.
- $P(n, r)$: choose $r$ items, then take all permutations of the items.

$$C(n, r) = P(n, r) / r!$$




## Recall from Last Week

$P(A)$ = the ***power set*** of $A$
= the set of all subsets of $A$

***Bijection between subsets and strings:***

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $A$ | = | $\{a_1,$ | $a_2,$ | $a_3,$ | $a_4,$ | $a_5,$ | $\ldots,$ | $a_n\}$ |
| String | = | 1 | 0 | 1 | 1 | 0 | … | 1 |
| Subset | = | $\{a_1,$ | | $a_3,$ | $a_4,$ | | $\ldots,$ | $a_n\}$ |

$$|P(A)| = 2^n$$




## Combinations and the Power Set

$$|P(A)| = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

*size-0 subsets*, *size-1 subsets*, *size-2 subsets*, …, *size-n subsets*

***Identity:*** $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

(Derive from the binomial theorem.)

## Poker: Ranks of Hands

| | | | |
|---|---|---|---|
| Straight Flush | = | 40 | $\approx 10^2$ |
| Four-of-a-Kind | = | 624 | $\approx 10^3$ |
| Full House | = | 3,744 | $\approx 10^4$ |
| Flush | = | 5,108 | $\approx 10^4$ |
| Straight | = | 10,240 | $\approx 10^4$ |
| Three-of-a-Kind | = | 54,912 | $\approx 10^5$ |
| Two Pairs | = | 123,552 | $\approx 10^5$ |
| One Pair | = | 1,098,240 | $\approx 10^6$ |
| High Card | = | 1,302,540 | $\approx 10^6$ |
| **Total** | = | **2,598,960** | $\approx \mathbf{10^6}$ |




## In-Class Problem 2