Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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In-Class Problems — Week 10, Fri

Problem 1. You are organizing a neighborhood census and instruct your census takers to knock on doors and note the sex of any child that answers the knock. Assume that there are two children in a household and that girls and boys are equally likely to be children and to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either B or G for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is E or Y indicating whether the *e*lder child or *y*ounger child opened the door. For example, $(\mathtt{B}, \mathtt{G}, \mathtt{Y})$ is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in T and O.

(b) What is the probability $\Pr\{T \mid O\}$, that both children are girls, given that a girl opened the door?

(c) Where is the mistake in the following argument?

> If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is
>
> $$1 - \Pr\{\text{both children are boys}\} = 1 - (1/2 \times 1/2) = 3/4. \quad (1)$$
>
> So,
>
> $$\begin{aligned} &\Pr\{T \mid \text{there is at least one girl in the household}\} && (2)\\ &= \frac{\Pr\{T \cap \text{there is at least one girl in the household}\}}{\Pr\{\text{there is at least one girl in the household}\}} && (3)\\ &= \frac{\Pr\{T\}}{\Pr\{\text{there is at least one girl in the household}\}} && (4)\\ &= (1/4)/(3/4) = 1/3. && (5)\end{aligned}$$
>
> Therefore, given that a girl opened the door, the probability that there are two girls in the household is 1/3.

(WE DIDN'T GET TO THE NEXT TWO PROBLEMS IN CLASS ON FRIDAY.)

Problem 2. Suppose there are 100 people in a room. Assume that their birthdays are independent and uniformly distributed. As stated in lecture notes, with probability $> 99\%$ there will be two that have the same birthday.

Now suppose you find out the birthdays of all the people in the room except one—call her "Jane"—and find all 99 dates to be different.

(a) What's wrong with the following argument:

> With probability greater than 99%, some pair of people in the room have the same birthday. Since the 99 people we asked all had different birthdays, it follows that with probability greater than 99% Jane has the same birthday as some other person in the room.

(b) What is the actual probability that Jane has the same birthday as some other person in the room?

Problem 3. Suppose you repeatedly flip a fair coin until you see the sequence HHT or the sequence HTT. What is the probability you see the sequence HTT before you see the sequence HHT? *Hint:* Try to find the probability that HHT comes before HTT conditioning on whether you first toss an H or a T. The answer is not $1/2$.

Probability Rules

Events

$$\Pr\left\{\bigcup_{n\in\mathbb{N}} A_n\right\} = \sum_{n\in\mathbb{N}} \Pr\{A_n\} \text{ for pairwise disjoint } A_n \qquad \text{(Sum Rule)}$$

$$\Pr\{A - B\} = \Pr\{A\} - \Pr\{A \cap B\} \qquad \text{(Difference Rule)}$$

$$\Pr\{\overline{B}\} = 1 - \Pr\{B\} \qquad \text{(Complement Rule)}$$

$$\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\} - \Pr\{A \cap B\} \qquad \text{(Inclusion-Exclusion)}$$

$$\Pr\{A \cup B\} \le \Pr\{A\} + \Pr\{B\} \qquad \text{(Boole's inequality)}$$

$$\Pr\{A\} \le \Pr\{A \cup B\} \qquad \text{(Monotonicity)}$$

[Law of Total Probability] Suppose the sample space is the disjoint union of $B_0, B_1, \ldots$. Then for all events A,

$$\Pr\{A\} = \sum_{i\in\mathbb{N}} \Pr\{A \cap B_i\}.$$

Conditional Probability

$$\Pr\{A \mid B\} ::= \frac{\Pr\{A \cap B\}}{\Pr\{B\}}$$

$$\Pr\{A \cap B\} = \Pr\{A \mid B\} \Pr\{B\} \qquad \text{(Product Rule)}$$

$$\Pr\{A \mid B\} = \frac{\Pr\{B \mid A\} \Pr\{A\}}{\Pr\{B\}} \qquad \text{(Bayes Rule)}$$

[Law of Total Probability - Conditional form] Suppose the sample space is the disjoint union of $B_0, B_1, \ldots$. Then for all events A,

$$\Pr\{A\} = \sum_{i \in \mathbb{N}} \Pr\{A \mid B_i\} \Pr\{B_i\}.$$

Independence

Definition. Events A and B are *independent* iff

$$\Pr\{A \cap B\} = \Pr\{A\} \Pr\{B\}.$$

Events $A_0, A_1, A_2, \ldots$ are *mutually independent* iff for all subsets $J \subset \mathbb{N}$,

$$\Pr\left\{\bigcap_{i \in J} A_i\right\} = \prod_{i \in J} \Pr\{A_i\}.$$

Events $A_0, A_1, A_2, \ldots$ are *k-wise independent* iff $\{A_i \mid i \in J\}$ are mutually independent for all subsets $J \subset \mathbb{N}$ with $|J| = k$.

Birthday Probabilities

Suppose there are s students and d days in the year. Let D be the event that all the students have ***d***ifferent birthdays. Then

$$\Pr\{D\} \sim e^{-(s^2/2d)}.$$

as long as $s = o(\sqrt[3]{d^2})$.